Abstract: This paper is concerned with machine repair system having on line units along with mixed standby units under the care of single unreliable server. Whenever any unit fails, it is immediately replaced by a standby unit if available. In case when all standbys are used, the failure of remaining on line units occurs in degraded fashion. The server is turned on when there are $N$ or more failed units present in the system; it is turned off when the system has no failed unit. The server is subject to breakdowns and repairs. The life time and repair time of failed units and the server are exponentially distributed. The governing equations are constructed using appropriate birth-death rates. After taking Laplace transform of set of equations, we employ matrix method to determine transient probabilities. Various performance indices such as expected number of standby units, expected number of failed units, probability of server being idle, busy or broken etc. are determined. Expressions for the system reliability and mean time to system failure are established in terms of transient probabilities. Cost function is also facilitated which may be helpful for system designer to achieve effective system descriptors. Numerical results are provided for various system characteristics to examine the effects of different parameters.

Keywords: Cost function, degraded failure, mixed standbys, $N$-policy, reliability, unreliable server.

1. Introduction

With the advancement of modern automated technology the importance of effectiveness of machines has been realized by specifying the optimal number of machines, spare part support and repair crew for the maintenance of the system. The performance of any machining system is highly influenced by server breakdown, which may also affect the system reliability indices. This problem may be realized in the area of computer, communication and transportation systems, etc. Production/manufacturing system may also not operate during the period of breakdown and this may lead to a loss of production.

Many researchers have studied the problem of unreliable server in different frameworks and suggested ways and means to tackle such situations. Initially, the concept of unreliable single server in queueing theory was introduced by White and Christie [28]. Sztrik and Bunday [21] studied single server Markovian queueing model with service interruption. Wang and Kuo [26] suggested profit analysis of machine repair problem with unreliable server. Grey et al. [3] incorporated the server breakdown for vacation queueing model. Grey et al. [4] extended their paper for queueing model with backup server and service interruption. Bilevel control for unreliable server was discussed by Ke [13]. A manufacturing system consisting of operating machines and spare machines with unpredictable breakdowns of repair facility was considered by Ke and Lin [12].

According to $N$-policy concept, the server is turned on when there are $N$ or more failed units in the system and turned off when there is no failed unit present in the system. Wang
provided optimal $N$-policy to study the optimal operation of single unreliable server in Markovian queueing model. Various characteristics have been discussed by Jain [8] for single server queue subject to breakdowns and repairs with dependent arrival rate under $N$-policy. Gupta [6] considered $N$-policy queueing system with finite source and warm spares. Optimal $N$-control policy was also discussed by Luh and Viniotis [16]. Pearn and Chang [18] established the cost formula under $N$-policy to minimize the total expected cost per customer per unit time. Recently Tian and Zhang [23] discussed the performance evaluation and optimization issues in vacation system with threshold policy. Parthasarathy and Sudhesh [17] suggested exact transient solution for the state probabilities of $M/M/c$ queueing system under $N$-policy.

To cope up with random failure of units, in many systems working in machining environment, whenever any unit fails, it is immediately replaced by standby unit if available. The provision of standbys can be helpful in smooth functioning of manufacturing/production. System standbys are assumed to be either cold or warm. Earlier Gross et al. [5] developed queueing model with spare. Sivazlian and Wang [20] considered machine repair model with warm standbys. Transient analysis of Markovian queueing model with spares was given by Jain and Dhyani [9]. Wang and Kuo [25] suggested cost strategy of series system with mixed standbys. In many systems when all standbys are exhausted, the remaining on line units may fail with degraded failure rate. Machine repair problem with degraded failure rate are closer to real time system. Goyal and Tantawi [2] evaluated performability of degradable computer system. Jain et al. [10] discussed bilevel control policy of degraded machining system with warm standbys. Profit analysis of the $M/M/R$ machine repair problem with two types of standbys was discussed by Wang et al. [27].

In order to have maximum profit with maximum possible efficiency of any system, the reliability of the system has to be increased. Introducing standbys we may increase the system reliability. Kumagi [14] analyzed the reliability of $n$-spare system with single server. Petrovic [19] established that the system reliability is improved by adding redundancy. Maximum reliability of the system was discussed by Amari et al. [1]. Reliability and cost optimization in distributed computing systems was suggested by Hsieh and Hsieh [7]. Lam et al. [15] derived some queueing characteristics of the system and reliability indices of the service station. Tadj [22] used matrix analytical solution to investigate $N$-policy in a queueing system. Jain et al. [11] also used matrix method to suggest $N$-policy for a machine repair system. The purpose of this investigation is to determine queueing performance measures and reliability indices in terms of transient probabilities for machine repair problem with mixed standbys and unreliable single server. We have employed matrix method to determine transient probabilities.

The rest of the paper is organized as follows. We describe the model along with assumptions and notations in section 2. The transient state equations using appropriate birth-death rates are constructed in section 3. The mathematical analysis is done in section 4. In section 5 expressions for queuing and reliability indices are established using queue size distribution. The cost function and optimal threshold parameters are evaluated in section 6. In order to validate the analytical results, we provide numerical results in section 7. The sensitivity analysis is also carried out. Finally conclusion is drawn in section 8.

2. Model Description

Consider mixed standby machining system with unreliable server. The following assumptions and notations are used to formulate the model:

- The system consists of $M$ online, $Y$ cold standby and $S$ warm standby units under the
In case of failure of an online unit, it is replaced by cold standby, if available, otherwise is replaced by warm standby. When all standbys are used, the system may also works till \( k \) operating units functioning well.

- The life time and repair time of online units, standbys and server are exponentially distributed.
- The server, which restores the failed units, turns on when \( N \) or more failed units are accumulated, it turns off when system becomes empty.
- Single unreliable server repairs the failed units in FCFS order.
- After repair, the failed units work as good as new.
- When all standbys are exhausted, the failure rate of remaining online units increase and the system is called to work in short mode.
- The switch over times to replace the failed units by standbys and repaired units to standbys are negligible.
- System fails when there are \( L = M + Y + S - k + 1 \) (\( k = 1, 2, \ldots, M \)) or more failed units in the system or server is in breakdown state.

For formulating the mathematical model, the following notations are used:

- \( M \): Total number of online units in the system.
- \( Y \): Total number of cold standbys in the system.
- \( S \): Total number of warm standbys in the system.
- \( \lambda \): Mean failure rate of online units.
- \( \lambda_d \): Degraded failure rate of remaining online units (\( \lambda_d \geq \lambda \)) when all standbys are exhausted and system works in degraded i.e. short mode.
- \( \nu \): Mean failure rate of warm standbys (\( \nu < \lambda \)).
- \( \mu \): Mean service rate of failed units.
- \( \alpha \): Failure rate of the server.
- \( \beta \): Repair rate of the server.

The state of the server at time \( t \) is denoted by variable \( \xi(t) \) as follows:

\[
\xi(t) = \begin{cases} 
0, & \text{when server is turned off.} \\
1, & \text{when server is turned on and busy in rendering repair of failed units.} \\
2, & \text{when server is turned on and broken down.}
\end{cases}
\]

Define the probabilities as

\[
P_{0,n}(t) : \text{Probability of } n \text{ failed units at time } t \text{ in the system when the server is turned off. (} n = 0, 1, \ldots, N - 1). \\
P_{i,n}(t) : \text{Probability of } n \text{ failed units at time } t \text{ in the system (} n = 1, 2, \ldots, L) \text{ i = 1, 2 denote that the server is in turned on and broken down states, respectively.}
\]

The Laplace transform of probabilities are defined as:

\[
\tilde{P}_{0,n}(s) = L\{P_{0,n}(t)\} = \int_0^\infty e^{-st} P_{0,n}(t)dt, \quad \tilde{P}_{i,n}(s) = L\{P_{i,n}(t)\} = \int_0^\infty e^{-st} P_{i,n}(t)dt.
\]
3. The Governing Equations

The effective mean failure rates which depends upon the states of the server, are defined as:

(i) When server is turned off \( i.e., \quad \xi(t) = 0 \)

\[
\lambda_{0,n} = M\lambda + Sv, \quad 0 \leq n < N.
\]

(ii) When server is turned on \( i.e., \quad \xi(t) = i \ (i = 1, 2) \)

\[
\lambda_{i,n} = \begin{cases} 
M\lambda + Sv, & 1 \leq n < Y \\
M\lambda + (Y + S - n)v, & Y \leq n < Y + S \\
(M + Y + S - n)\lambda_{i-1}, & Y + S \leq n < L \\
0 & \text{otherwise.}
\end{cases}
\]

The differential difference equations are constructed by taking appropriate transition rates as follows:

\[
P'_{0,0}(t) = -\lambda_{0,0}P_{0,0}(t) + \mu P_{1,1}(t),
\]

\[
P'_{0,n}(t) = -\lambda_{0,n}P_{0,n}(t) + \lambda_{0,n-1}P_{0,n-1}(t), 1 \leq n \leq N - 1,
\]

\[
P'_{1,1}(t) = -\lambda_{1,1} + \mu + \alpha)P_{1,1}(t) + \mu P_{1,2}(t) + \beta P_{2,1}(t),
\]

\[
P'_{1,n}(t) = -\lambda_{1,n} + \mu + \alpha)P_{1,n}(t) + \lambda_{1,n-1}P_{1,n-1}(t) + \mu P_{1,n+1}(t) + \beta P_{2,n}(t), 2 \leq n < N,
\]

\[
P'_{1,N}(t) = -\lambda_{1,N} + \mu + \alpha)P_{1,N}(t) + \lambda_{1,N-1}P_{1,N-1}(t) + \mu P_{1,N+1}(t) + \beta P_{2,N}(t) + \lambda_{0,N-1}P_{0,N-1}(t),
\]

\[
P'_{1,n}(t) = -\lambda_{1,Y} + \mu + \alpha)P_{1,Y}(t) + \lambda_{1,Y-1}P_{1,Y-1}(t) + \mu P_{1,Y+1}(t) + \beta P_{2,Y}(t),
\]

\[
P'_{1,Y+1}(t) = -\lambda_{1,Y+1} + \mu + \alpha)P_{1,Y+1}(t) + \lambda_{1,Y}P_{1,Y}(t) + \mu P_{1,Y+2}(t) + \beta P_{2,Y+1}(t),
\]

\[
P'_{1,n}(t) = -\lambda_{1,n} + \mu + \alpha)P_{1,n}(t) + \lambda_{1,n-1}P_{1,n-1}(t) + \mu P_{1,n+1}(t) + \beta P_{2,n}(t), Y + 1 < n < Y + S,
\]

\[
P'_{1,Y+S}(t) = -\lambda_{1,Y+S} + \mu + \alpha)P_{1,Y+S}(t) + \lambda_{1,Y+S-1}P_{1,Y+S-1}(t) + \mu P_{1,Y+S+1}(t) + \beta P_{2,Y+S}(t),
\]

\[
P'_{1,Y+S+1}(t) = -\lambda_{1,Y+S+1} + \mu + \alpha)P_{1,Y+S+1}(t) + \lambda_{1,Y+S}P_{1,Y+S}(t) + \mu P_{1,Y+S+2}(t) + \beta P_{2,Y+S+1}(t),
\]

\[
P'_{1,n}(t) = -\lambda_{1,n} + \mu + \alpha)P_{1,n}(t) + \lambda_{1,n-1}P_{1,n-1}(t) + \mu P_{1,n+1}(t) + \beta P_{2,n}(t), Y + S + 2 \leq n < L - 1,
\]

\[
P'_{1,L-1}(t) = -\lambda_{1,L-1} + \mu + \alpha)P_{1,L-1}(t) + \lambda_{1,L}P_{1,L}(t) + \mu P_{1,L}(t) + \beta P_{2,L-1}(t),
\]

\[
P'_{2,1}(t) = -\lambda_{2,1} + \beta)P_{2,1}(t) + \alpha P_{1,1}(t),
\]

\[
P'_{2,n}(t) = -\lambda_{2,n} + \beta)P_{2,n}(t) + \alpha P_{1,n}(t) + \lambda_{2,n-1}P_{2,n-1}(t), 2 \leq n < N,
\]

\[
P'_{2,N}(t) = -\lambda_{2,N} + \beta)P_{2,N}(t) + \alpha P_{1,N}(t) + \lambda_{2,N-1}P_{2,N-1}(t),
\]
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\[ P'_{2,n}(t) = -(\lambda_{2,n} + \beta)P_{2,n}(t) + \alpha P_{1,n}(t) + \lambda_{2,n-1}P_{2,n-1}(t), \quad N + 1 \leq n < Y, \]  
(18)

\[ P'_{2,Y}(t) = -(\lambda_{2,Y} + \beta)P_{2,Y}(t) + \alpha P_{1,Y}(t) + \lambda_{2,Y-1}P_{2,Y-1}(t), \]  
(19)

\[ P'_{2,Y+1}(t) = -(\lambda_{2,Y+1} + \beta)P_{2,Y+1}(t) + \alpha P_{1,Y+1}(t) + \lambda_{2,Y}P_{2,Y}(t), \]  
(20)

\[ P'_{2,n}(t) = -(\lambda_{2,n} + \beta)P_{2,n}(t) + \alpha P_{1,n}(t) + \lambda_{2,n-1}P_{2,n-1}(t), \quad Y + 1 < n < Y + S, \]  
(21)

\[ P'_{2,Y+S}(t) = -(\lambda_{2,Y+S} + \beta)P_{2,Y+S}(t) + \alpha P_{1,Y+S}(t) + \lambda_{2,Y+S-1}P_{2,Y+S-1}(t), \]  
(22)

\[ P'_{2,Y+S+1}(t) = -(\lambda_{2,Y+S+1} + \beta)P_{2,Y+S+1}(t) + \alpha P_{1,Y+S+1}(t) + \lambda_{2,Y+S}P_{2,Y+S}(t), \]  
(23)

\[ P'_{2,n}(t) = -(\lambda_{2,n} + \beta)P_{2,n}(t) + \alpha P_{1,n}(t) + \lambda_{2,n-1}P_{2,n-1}(t), \quad Y + S + 2 \leq n < L - 1, \]  
(24)

\[ P'_{2,L-1}(t) = -(\lambda_{2,L-1} + \beta)P_{2,L-1}(t) + \alpha P_{1,L-1}(t) + \lambda_{2,L-2}P_{2,L-2}(t), \]  
(25)

\[ P'_{2,L}(t) = -\beta P_{2,L}(t) + \alpha P_{1,L}(t) + \lambda_{2,L-1}P_{2,L-1}(t), \]  
(26)

The initial conditions are as:

\[ P_{0,n}(0) = \begin{cases} 1, & n = 0 \\ 0, & n = 1, 2, ..., N - 1 \end{cases} \quad \text{and} \quad P_{n,n}(0) = 0, \quad n = 1, 2, ..., L. \]

The Laplace transforms of equations (1)-(26), provide

\[ (s + \lambda_{0,0})\tilde{P}_{0,0}(s) - \mu \tilde{P}_{1,1}(s) = P_{0,0}(0), \]  
(27)

\[ (s + \lambda_{0,n})\tilde{P}_{0,n}(s) - \lambda_{0,n-1}\tilde{P}_{0,n-1}(s) = P_{0,n}(0), \quad 1 \leq n \leq N - 1, \]  
(28)

\[ (s + \lambda_{1,1} + \mu + \alpha)\tilde{P}_{1,1}(s) - \mu \tilde{P}_{1,2}(s) - \beta \tilde{P}_{2,1}(s) = P_{1,1}(0), \]  
(29)

\[ (s + \lambda_{1,n} + \mu + \alpha)\tilde{P}_{1,n}(s) - \lambda_{1,n-1}\tilde{P}_{1,n-1}(s) - \mu \tilde{P}_{1,n+1}(s) - \beta \tilde{P}_{2,n}(s) = P_{1,n}(0), \quad 2 \leq n < N, \]  
(30)

\[ (s + \lambda_{N,n} + \mu + \alpha)\tilde{P}_{N,n}(s) - \lambda_{N-1,1}\tilde{P}_{N-1,1}(s) - \mu \tilde{P}_{N,n+1}(s) - \beta \tilde{P}_{2,N}(s) = P_{N,n}(0), \]  
(31)

\[ (s + \lambda_{1,n} + \mu + \alpha)\tilde{P}_{1,n}(s) - \lambda_{1,n-1}\tilde{P}_{1,n-1}(s) - \mu \tilde{P}_{1,n+1}(s) - \beta \tilde{P}_{2,n}(s) = P_{1,n}(0), \quad N + 1 \leq n < Y, \]  
(32)

\[ (s + \lambda_{1,Y} + \mu + \alpha)\tilde{P}_{1,Y}(s) - \lambda_{1,Y-1}\tilde{P}_{1,Y-1}(s) - \mu \tilde{P}_{1,Y+1}(s) - \beta \tilde{P}_{2,Y}(s) = P_{1,Y}(0), \]  
(33)

\[ (s + \lambda_{1,Y+1} + \mu + \alpha)\tilde{P}_{1,Y+1}(s) - \lambda_{1,Y}\tilde{P}_{1,Y}(s) - \mu \tilde{P}_{1,Y+2}(s) - \beta \tilde{P}_{2,Y+1}(s) = P_{1,Y+1}(0), \]  
(34)

\[ (s + \lambda_{1,n} + \mu + \alpha)\tilde{P}_{1,n}(s) - \lambda_{1,n-1}\tilde{P}_{1,n-1}(s) - \mu \tilde{P}_{1,n+1}(s) - \beta \tilde{P}_{2,n}(s) = P_{1,n}(0), \quad Y + 1 < n < Y + S, \]  
(35)

\[ (s + \lambda_{1,Y+S} + \mu + \alpha)\tilde{P}_{1,Y+S}(s) - \lambda_{1,Y+S-1}\tilde{P}_{1,Y+S-1}(s) - \mu \tilde{P}_{1,Y+S+1}(s) - \beta \tilde{P}_{2,Y+S}(s) = P_{1,Y+S}(0), \]  
(36)

\[ (s + \lambda_{1,Y+S+1} + \mu + \alpha)\tilde{P}_{1,Y+S+1}(s) - \lambda_{1,S}\tilde{P}_{1,Y+S}(s) - \mu \tilde{P}_{1,Y+S+2}(s) - \beta \tilde{P}_{2,Y+S+1}(s) = P_{1,Y+S+1}(0), \]  
(37)

\[ (s + \lambda_{1,n} + \mu + \alpha)\tilde{P}_{1,n}(s) - \lambda_{1,n-1}\tilde{P}_{1,n-1}(s) - \mu \tilde{P}_{1,n+1}(s) - \beta \tilde{P}_{2,n}(s) = P_{1,n}(0), \quad Y + S + 2 \leq n < L - 1, \]  
(38)
(s + \lambda_{1,L-1} + \mu + \alpha) \tilde{P}_{1,L-1}(s) - \lambda_{1,L-2} \tilde{P}_{1,L-2}(s) - \mu \tilde{P}_{1,L}(s) - \beta \tilde{P}_{2,L-1}(s) = P_{1,L-1}(0), \quad (39)

(s + \lambda_{2,1} + \beta) \tilde{P}_{2,1}(s) - \lambda_{2,2-1} \tilde{P}_{2,2-1}(s) = P_{2,1}(0), \quad (40)

(s + \lambda_{2,2-1} + \beta) \tilde{P}_{2,2-1}(s) = P_{2,2-1}(0), \quad (41)

(s + \lambda_{2,n} + \beta) \tilde{P}_{2,n}(s) - \alpha \tilde{P}_{1,n}(s) = P_{2,n}(0), \quad 2 \leq n < N, \quad (42)

(s + \lambda_{2,N} + \beta) \tilde{P}_{2,N}(s) - \alpha \tilde{P}_{1,N}(s) = P_{2,N}(0), \quad (43)

(s + \lambda_{2,1} + \beta) \tilde{P}_{2,1}(s) - \lambda_{2,2-1} \tilde{P}_{2,2-1}(s) = P_{2,1}(0), \quad (44)

(s + \lambda_{2,Y} + \beta) \tilde{P}_{2,Y}(s) - \alpha \tilde{P}_{1,Y}(s) = P_{2,Y}(0), \quad (45)

(s + \lambda_{2,Y+1} + \beta) \tilde{P}_{2,Y+1}(s) = P_{2,Y+1}(0), \quad (46)

(s + \lambda_{2,Y} + \beta) \tilde{P}_{2,Y}(s) - \alpha \tilde{P}_{1,Y}(s) = P_{2,Y}(0), \quad (47)

(s + \lambda_{2,Y+S} + \beta) \tilde{P}_{2,Y+S}(s) - \alpha \tilde{P}_{1,Y+S}(s) = P_{2,Y+S}(0), \quad (48)

(s + \lambda_{2,Y+S+1} + \beta) \tilde{P}_{2,Y+S+1}(s) = P_{2,Y+S+1}(0), \quad (49)

(s + \lambda_{2,n} + \beta) \tilde{P}_{2,n}(s) - \alpha \tilde{P}_{1,n}(s) = P_{2,n}(0), \quad (50)

(s + \lambda_{2,L-1} + \beta) \tilde{P}_{2,L-1}(s) - \alpha \tilde{P}_{1,L-1}(s) = P_{2,L-1}(0), \quad (51)

(s + \beta) \tilde{P}_{2,L}(s) = P_{2,L}(0), \quad (52)

4. The Analysis

Equations (27)-(52) can be put in matrix form as:

\[ B(s) \tilde{P}(s) = P(0), \quad (53) \]

where

\[
B(s) = \begin{bmatrix} A_1 & G_1 & 0 \\ G_2 & A_2 & C_1 \\ 0 & C_2 & A_3 \end{bmatrix}_{(N+2L) \times (N+2L)}
\]

\[
A_i = \begin{bmatrix} s + \lambda_{0,0} & 0 & 0 & \ldots & 0 & 0 \\ -\lambda_{0,0} & s + \lambda_{0,1} & 0 & \ldots & 0 & 0 \\ 0 & -\lambda_{0,1} & s + \lambda_{0,2} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & s + \lambda_{0,N-2} & 0 \\ 0 & 0 & 0 & \ldots & -\lambda_{0,N-2} & s + \lambda_{0,N-1} \end{bmatrix}_{(N \times N)}
\]
\[
A_2 = \begin{bmatrix}
(s + \lambda_{1}^1 + \alpha) & -\mu & 0 & \ldots & 0 & 0 \\
-\lambda_{1}^1 & (s + \lambda_{1}^2 + \alpha) & -\mu & \ldots & 0 & 0 \\
0 & -\lambda_{1}^2 & (s + \lambda_{1}^3 + \alpha) & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & (s + \lambda_{1,L-1} + \alpha) & -\mu \\
0 & 0 & 0 & \ldots & -\lambda_{1,L-1} & (s + \lambda + \alpha)
\end{bmatrix}_{(L \times L)}
\]

\[
A_3 = \begin{bmatrix}
(s + \lambda_{2}^1 + \beta) & 0 & 0 & \ldots & 0 & 0 \\
-\lambda_{2}^1 & (s + \lambda_{2}^2 + \beta) & 0 & \ldots & 0 & 0 \\
0 & -\lambda_{2}^2 & (s + \lambda_{2}^3 + \beta) & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & (s + \lambda_{2,L-1} + \beta) & 0 \\
0 & 0 & 0 & \ldots & -\lambda_{2,L-1} & (s + \beta)
\end{bmatrix}_{(L \times L)}
\]

\[C_1 = -\beta \mathbf{I}, \quad C_2 = -\alpha \mathbf{I}, \quad \text{where } \mathbf{I} \text{ is identity matrix of order } L \times L.\]

Also \( \bar{P}(s) = [\bar{P}_{0,n}(s), \bar{P}_{i,n}(s), \bar{P}_{i,n}(s)] \) is a column vector of order \((N + 2L) \times 1\) and

\[
\bar{P}_{0,n}(s) = [\bar{P}_{0,0}(s), \bar{P}_{0,1}(s), \ldots, \bar{P}_{0,N-1}(s)]_{N \times 1},
\]

\[
\bar{P}_{i,n}(s) = [\bar{P}_{i,1}(s), \bar{P}_{i,2}(s), \ldots, \bar{P}_{i,N-1}(s), \bar{P}_{i,N}(s), \bar{P}_{i,N+1}(s), \ldots, \bar{P}_{i,L-1}(s), \bar{P}_{i,L}(s)]_{L \times 1}, \quad \text{for } i = 1, 2,
\]

\(P(0) = [1, 0, 0, \ldots, 0, 0, 0, \ldots, 0, 0, 0, 0, \ldots, 0]_{(N + 2L) \times 1} \) is an initial vector.

To compute probabilities \( \bar{P}_{i,n}(s), \quad (i = 0, 1, 2), \) we apply Cramer’s rule on matrix \( B(s) \) and obtain

\[
\bar{P}_{i,n}(s) = \frac{\Delta B'_{n+1}(s)}{\Delta B(s)},
\]

where \( \Delta B(s) \) is the determinant of the matrix \( B(s) \) and matrix \( B'_{n+1}(s) \) is obtained by replacing:
(i) \((n+1)\)th column of \(B(s)\) with initial vector \(P(0)\) for \(i = 0; n = 0, 1, \ldots, N-1.\)

(ii) \((N+n)\)th column of \(B(s)\) with initial vector \(P(0)\) for \(i = 1; n = 1, 2, \ldots, L.\)

(iii) \((N+L+n)\)th column of \(B(s)\) with initial vector \(P(0)\) for \(i = 2; n = 1, 2, \ldots, L.\)

Now we calculate characteristic roots of tridiagonal matrix \(B(s)\). \(s = 0\) is one of the roots.

Let \(s = -d\), so that we obtain \(B(-d) = (B - dI)\).

Now equation (53) becomes

\[
B(-d) \tilde{P}(s) = (B - dI) \tilde{P}(s) = P(0).
\]

Other \((N + 2L - 1)\) roots in which \(r\) are real roots and \(\ell\) are complex roots in pairs are denoted by:

\[
d_1, d_2, \ldots, d_r \text{ and } (d_{r+1}, \bar{d}_{r+1}), (d_{r+2}, \bar{d}_{r+2}), \ldots, (d_{r+\ell}, \bar{d}_{r+\ell}),
\]

respectively. Thus we have

\[
\Delta B(s) = s \left[ \prod_{k=1}^{r} (s + d_k) \right] \left[ \prod_{k=1}^{\ell} (s + d_{r+k}) (s + \bar{d}_{r+k}) \right]. \tag{55}
\]

Equations (54) and (55) yield

\[
\tilde{P}_{r,\ell}(s) = \frac{\Delta B'_{r+\ell}(s)}{s \left[ \prod_{k=1}^{r} (s + d_k) \right] \left[ \prod_{k=1}^{\ell} (s + d_{r+k}) (s + \bar{d}_{r+k}) \right]}.
\]

On expanding by partial fractions

\[
= \frac{a_0}{s} + \frac{a_1}{s + d_1} + \ldots + \frac{a_r}{s + d_r} + \frac{b_1 s + c_1}{(s + d_{r+1})(s + \bar{d}_{r+1})} + \ldots + \frac{b_\ell s + c_\ell}{(s + d_{r+\ell})(s + \bar{d}_{r+\ell})}. \tag{56}
\]

Here \(a_0\) and \(a_m (m = 1, 2, \ldots, r)\) are real numbers calculated as:

\[
a_0 = \frac{\Delta B'_{r+1}(0)}{\left( \prod_{k=1}^{r} d_k \right) \left( \prod_{k=1}^{\ell} \bar{d}_{r+k} \right)}, \tag{57}
\]

\[
a_m = \frac{\Delta B'_{r+\ell+1}(-d_m)}{\left( \prod_{k=1}^{r} (d_k - d_m) \right) \left( \prod_{k=m}^{\ell} (-d_m + d_{r+k}) (-d_m + \bar{d}_{r+k}) \right)}, \tag{58}
\]

\[
b_m (-d_{r+m}) + c_m = \frac{\Delta B'_{r+\ell+1}(-d_{r+m})}{\left( \prod_{k=1}^{r} (d_k - d_{r+m}) \right) \left( \prod_{k=m}^{\ell} (-d_{r+m} + d_{r+k}) (-d_{r+m} + \bar{d}_{r+k}) \right)}, \tag{59}
\]

\((m = 1, 2, \ldots, \ell)\).
Let complex characteristic root $r_m$ be a combination of real part $u_m$ and imaginary part $v_m$. On taking inverse Laplace transform of equation (56), we get

$$P_{i,n}(t) = a_0 + \sum_{m=1}^{M} a_m e^{-u_m t} + \sum_{m=1}^{M} \left[ b_m e^{-u_m t} \cos(v_m t) + \frac{c_m - b_m u_m}{v_m} e^{-u_m t} \sin(v_m t) \right],$$

where $a_0, a_m, b_m, c_m, u_m$ and $v_m$ are all real numbers.

5. Performance Indices

After determining the probabilities for different system states, now we provide expressions for various performance indices as follows:

(I) Expected number of cold standbys units in the system at time $t$

$$E\{Y(t)\} = \sum_{i=0}^{k} \sum_{n=0}^{Y-i} (Y-n) P_{i,n}(t).$$

(II) Expected number of warm standbys in the system at time $t$

$$E\{S(t)\} = S \sum_{i=0}^{k} \sum_{n=0}^{Y+S-i} P_{i,n}(t) + \sum_{i=0}^{k} \sum_{n=Y+1}^{Y+S} (Y+S-n) P_{i,n}(t).$$

(III) Expected number of short units when all standbys are exhausted at time $t$

$$E\{X(t)\} = \sum_{i=1}^{k} \sum_{n=0}^{Y+S-i} (n-(Y+S)) P_{i,n}(t).$$

(IV) Expected number of failed units in the queue at time $t$

$$E\{N_q(t)\} = \sum_{n=0}^{N-1} n P_{0,n}(t) + \sum_{i=1}^{L} \sum_{n=0}^{Y-i} (n-1) P_{i,n}(t).$$

(V) Expected number of failed units in the system at time $t$

$$E\{N(t)\} = \sum_{n=0}^{N-1} n P_{0,n}(t) + \sum_{i=1}^{L} \sum_{n=1}^{Y-i} n P_{i,n}(t).$$

(VI) The probability of the server being idle at time $t$

$$P\{I(t)\} = \sum_{n=0}^{N-1} P_{0,n}(t).$$

(VII) The probability of the server being busy at time $t$

$$P\{B(t)\} = \sum_{n=1}^{L} P_{i,n}(t).$$

(VIII) The probability of the server being broken down and under repair at time $t$

$$P\{BD(t)\} = 1 - \left[ P\{I(t)\} + P\{B(t)\} \right].$$

(IX) Reliability of the system at time $t$ is given by

$$R(t) = \sum_{n=0}^{N-1} P_{0,n}(t) + \sum_{i=1}^{L-1} P_{i,n}(t).$$
6. Cost Function

In order to determine optimal threshold parameters, we construct a cost function as follows:

Average total cost per unit time is calculated as:
\[
TC(t; N, Y, S) = C_N E\{N(t)\} + C_X E\{X(t)\} + C_Y E\{Y(t)\} + C_S E\{S(t)\} + C_I P\{I(t)\} \\
+ C_B P\{B(t)\} + C_R P\{BD(t)\},
\]

where
\[
C_N = \text{Cost of repairing failed unit per unit time} \\
C_X = \text{Cost of repairing short unit per unit time} \\
C_Y = \text{Cost of providing each cold standby per unit time} \\
C_S = \text{Cost of providing each warm standby per unit time} \\
C_I = \text{Cost of server per unit time in idle state} \\
C_B = \text{Cost of server per unit time in busy state} \\
C_R = \text{Cost of repairing of the server per unit time}
\]

Now we formulate the cost minimization problem with reliability constraint as follows:
\[
z = \text{Minimize } TC(t; N, Y, S), \quad \text{(72)}
\]

Subject to \( R(t) \geq R_m(t) \). \quad \text{(73)}

Here \( R_m(t) \) denotes the minimum reliability required at time \( t \). To obtain optimal number of threshold parameter \( (N^*) \), and optimal number of standbys \( (Y^*, S^*) \), we use direct search method on substituting successive values of these parameters into the cost function until the minimum value of \( TC(t; N, Y, S) \), say \( TC^* \) which satisfying the above constraint (72) is obtained.

7. Numerical Results

Extensive numerical experiment has been done to study the effect of various system parameters on performance indices. The computer program has been prepared in software MATLAB and run on Pentium IV to implement the computational procedure suggested for the queueing and reliability indices.

To evaluate optimal threshold parameter \( (N^*) \), we summarize the numerical results for the average cost per unit time at different time \( t \) in tables 1(a) and 1(b) and plot 3d-graphs in figures 1(a) and 1(b) for different sets of cost elements, respectively by considering the default parameters fixed as \( M = 15, L = 15, Y = 5, S = 4, \lambda = .5, \lambda_s = .8, \nu = .2, \alpha = .03, \beta = .05, \mu = 15 \). We observe that initially cost decreases with respect to threshold parameter \( N \) and after a threshold value \( (N^*) \) of \( N \), it increases. After some
For fixed parameters \( M = 15, \ t = 5, \ L = 15, \ \lambda_d = 1.1, \ \nu = 2, \ C_N = 10, \ C_X = 20, \ C_Y = 20, \ C_S = 40, \ C_I = 40, \ C_B = 80, \ C_R = 50, \ R_m(t) = .80 \), Table 2 summarizes the results for the optimal values of threshold parameters \( (N^*) \) and both types of spares \( (Y^*, S^*) \), minimum average cost per unit time \( TC^* \) and reliability \( R(t) \) for different values of \( \lambda, (\alpha, \beta) \) and \( \mu \). Significant changes have been noted in minimum cost per unit time and optimal parameters \( (N^*, Y^*, S^*) \) by varying different parameters.
Based on numerical experiments we have shown the tractability of the computational procedure using direct search method.

Table 2. Optimal numbers of threshold parameter and standbys at minimum average cost per unit time.

<table>
<thead>
<tr>
<th>(α, β)</th>
<th>λ</th>
<th>μ = 8</th>
<th>μ = 10</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>N*, Y*, S*</td>
<td>TC*</td>
<td>R(t)</td>
</tr>
<tr>
<td>(.01,.03)</td>
<td>0.2</td>
<td>3, 3, 3</td>
<td>238.82</td>
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<td></td>
<td>0.3</td>
<td>3, 4, 3</td>
<td>266.15</td>
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<tr>
<td></td>
<td>0.4</td>
<td>3, 5, 3</td>
<td>285.62</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>7, 7, 4</td>
<td>292.87</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>8, 8, 5</td>
<td>282.95</td>
</tr>
<tr>
<td>(.03,.05)</td>
<td>0.2</td>
<td>3, 3, 3</td>
<td>241.76</td>
</tr>
<tr>
<td></td>
<td>0.3</td>
<td>3, 4, 3</td>
<td>269.01</td>
</tr>
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<td></td>
<td>0.4</td>
<td>8, 8, 3</td>
<td>292.3</td>
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<td>0.5</td>
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<tr>
<td></td>
<td>0.6</td>
<td>8, 8, 5</td>
<td>279.93</td>
</tr>
<tr>
<td>(.05,.07)</td>
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<td>4, 4, 3</td>
<td>253.23</td>
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<tr>
<td></td>
<td>0.3</td>
<td>3, 3, 3</td>
<td>281.95</td>
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<tr>
<td></td>
<td>0.6</td>
<td>8, 8, 5</td>
<td>277.26</td>
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</table>

Figures 1(a) and 1(b). Average total cost per unit time by varying threshold parameter (N) at different time t.

8. Conclusion

In this investigation we have obtained transient state probabilities for the multi component machine repair problem with unreliable server. The provision of mixed standbys is taken into account to facilitate the uninterrupted renewal of failed units as high reliable system is needed in many embedded real time systems namely electronic, computer,
communication, flexible manufacturing systems, etc. The mixed standby part support may play important role in particular when there is significant difference in the cost and size of the cold and warm standbys; an optimal combination of both should be recommended in such a case. The cost analysis for the system is also facilitated in order to determine the optimal threshold parameters.

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References


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