Comparison of Availability Between Two Systems with Warm Standby Units and Different Imperfect Coverage

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Abstract: This paper deals with two availability systems with warm standby units and different imperfect coverage. The time-to-failure and the time-to-repair of the active and standby units are assumed to be exponentially and generally distributed, respectively. We assume that the coverage factor of the active-unit failure is different from that of the standby-unit failure. We provide a recursive method, using the supplementary variable technique and treating the supplementary variable as the remaining repair time, to develop the steady-state availability, \( \text{Av} \), for two systems. An efficient Maple computer program is utilized to calculate the availability of two systems. Comparisons are performed for five different repair time distributions such as exponential, gamma, uniform, deterministic, and normal.

Keywords: Availability, comparisons, imperfect coverage, supplementary variable, warm standby units.

1. Introduction

Uncertainty is one of the important issues in management decisions. Two of the most useful uncertainty measures are: system reliability and system availability. Achieving a high or required level of availability is often an essential requisite. This paper deals with a supplementary variable technique to study the availability analysis of two models with warm standby units and different imperfect coverage. It may be impossible to switch in an existing spare module and then recover from a failure. Faults such as these are called to be not covered. We assume that the effect of failure of the active unit is different from that of the standby unit. Let the probabilities of successful recovery on the failure of an active unit and standby unit be denoted by \( c \) and \( c_s \), respectively. Quantity \( c \) (or \( c_s \)) which including the probabilities of successful detection, location, and recovery from a failure is known as the coverage factor or coverage probability (see Trivedi [13]). A standby unit is called a “warm standby” if its failure rate is nonzero and is less than the failure rate of an active unit. Active and warm standby units can be considered to be repairable.

The concept of coverage and its effect on the reliability and/or availability model of a repairable system has been introduced by several authors such as Amari et al. [1, 2], Arnold [3], Dugan and Trivedi [5], Trivedi [13], and etc. Amari et al. [1] provided a simple and efficient algorithm to compute reliability and unreliability of systems which are possibly subject to imperfect fault coverage. The status and trends of imperfect coverage models which associated reliability analysis techniques were introduced in Amari et al. [2]. The Doyle et al. [4] algorithm combines a coverage model with a combinatorial model to compute system unreliability. Hsu et al. [6] used a Bayesian approach to evaluate system characteristics of a repairable system with imperfect coverage and reboot. Based on four feasible and efficient approaches, Ke and Chu [7] performed computational comparisons of confidence intervals.
for the steady-state availability of a repairable system. Ke et al. [8] adopted a Bayesian approach to study system characteristics of a repairable system with detection, imperfect coverage, and reboot. Recently, Ke et al. [9] investigated asymptotic confidence limits for the steady-state availability, failure frequency, and mean time to failure of a two-unit redundant system with detection delay and imperfect coverage. The statistical inferences of an availability system with general repair distribution and imperfect fault coverage was analyzed by Ke et al. [10]. Levitin and Amari [11] introduced multi-state systems with multi-fault coverage where the effectiveness of recovery mechanism depends on the coexistence of multiple faults in related elements. The assessment of the reliability of redundant systems with imperfect fault coverage was studied by Myers and Rauzy [12]. Wang and Chiu [14] presented the cost benefit analysis of availability systems with warm standby units and imperfect coverage. They assumed that the coverage factor is the same for active and standby unit failures. The problem considered in this paper is more general than the works of Trivedi [13] and Wang and Chiu [14]. Wang and Chen [15] performed comparative analysis of availability between three systems with general repair times, reboot delay and switching failures. The explicit expression for the steady-state availability, $Av$, to two availability systems with different imperfect coverage and repair time distribution of the general type has not been found. We first present a recursive method, using the supplementary variable technique and treating the supplementary variable as the remaining repair time, to develop the steady-state availability for two systems. Next, we use an efficient computer program to calculate the steady-state availability of two systems. Finally, based on assumed numerical values given to the system parameters, comparisons are performed for five different repair time distributions such as exponential, gamma, uniform, deterministic, and normal.

2. Description of the System

We consider two availability systems as follows. The first system is the two-unit system but consider one unit as active and the other as warm standby unit. The second system is the three-unit system but consider one unit as active and two units as warm standbys.

We assume that standby generators are allowed to fail while inactive before they are put into full operation, and that the standby generators are continuously monitored by a fault detecting device in order to identify if they fail or not. Active units and warm standby units can be considered to be repairable. Each of the active units fails independently of the state of the others and has an exponential time-to-failure distribution with parameter $\lambda$. Whenever one of these units fails, it is immediately replaced by a warm standby unit if any is available. We now assume that each of the available standby units fails independently of the state of all the others and has an exponential time-to-failure distribution with parameter $\alpha$ ($0 < \lambda < \alpha$). When an active unit fails, it may be immediately detected, located, and replaced with a coverage probability $c$ by a standby if one is available. It is assumed that the replacing time is instantaneous. We assume that the effect of failure of the active unit is different from that of the standby unit. The failure of the standby unit while the active unit is still working is detected immediately with probability $c_*$. We define the unsafe failure state of the system as any one of the breakdowns is not covered. We continue with the assumption that active-unit failure (or standby-unit failure) in the unsafe failure state is cleared by a reboot. Reboot delay takes place at rate $\beta$ for an active unit (or standby unit) which is exponentially distributed.

The system fails when the standby units are emptied for which we define as the state of safe failure. It is assumed that the times to repair of the units are independent and identically distributed (i.i.d.) random variables having a distribution $B(u)$ ($u \geq 0$), a probability density function $b(u)$ ($u \geq 0$) and mean repair time $b_i$. If one active unit or standby unit is in repair, then arriving failed units have to wait in the queue until the server is available. Let us assume
that failed units arriving at the server form a single waiting line and are served in the order of their arrivals; i.e., according to the first-come, first-served discipline. Suppose that the server can serve only one active unit (or warm standby unit) at a time, and that the service is independent of the arrival of the units. Once a unit is repaired, it is as good as new.

3. Practical Justification of the Model

A practical problem related to the computer system is presented for illustrative purposes. We consider a Linux cluster which provides the web service and is managed by the Linux Virtual Server (LVS) cluster management system. The LVS director (treated as the active unit) receives requests from the Internet and forwards them to backend machines. The high availability of the cluster can be provided by deploying a backup server (treated as the warm standby unit) which running the fake software and a fault detecting server which running the mon software. The mon is a general-purpose monitoring system, which can be used to monitor network system availability and server nodes. Fake is an IP take-over software by using of Address Resolution Protocol (ARP) spoofing. The architecture of this high availability cluster is illustrated in Figure 1.

![Figure 1. A high availability Linux cluster providing web service.](image_url)

Initially both the director and backup server are working. The director fails independently of the state of the backup server and vice versa. Let the time-to-failure of the director and the time-to-failure of the backup server be exponentially distributed with parameter $\lambda$ and $\alpha$, respectively. When the director fails, it may be immediately detected by the fault detecting server, and replaced with a coverage probability $c$ by the backup server if it is available. With probability $1-c$ the fault detecting server fails to cover the failure of the director. The failure of the backup server while the director is still working is detected immediately with probability $c$. The Linux cluster is in the unsafe failure state as any one of the breakdowns is not covered. A director failure (or backup server failure) in the unsafe failure state can be cleared by a reboot. Reboot delay takes place at rate $\beta$ for a director (or backup server) which is exponentially distributed.
4. Availability Analysis of the First System

For the first system, we consider a two-unit system which consists of one active unit and one warm standby unit. The state-transition-rate diagram of the first system is shown in Figure 2. For further detail descriptions, we refer the reader to a textbook (Trivedi [13, p. 469]). We use the supplementary variable: $U \equiv$ remaining repair time for the unit under repair. The state of the system at time $t$ is given by

\[
N(t) \equiv \text{number of working units in the system}, \quad U(t) \equiv \text{remaining repair time for the unit being repaired.}
\]

Let us define

\[
P_{m,n}(u,t) dt = \Pr \{ N(t) = n, u < U(t) \leq u + du \}, \quad u \geq 0
\]

\[
P_{m,n}(u,t) dt = \int_{0}^{\infty} P_{m,n}(u,t) du,
\]

where $m$ denotes the number of active units and $n$ is the number of warm standby units.

![State-transition-rate diagram for the first system](image)

Figure 2. State-transition-rate diagram for the first system

Referring to the state-transition-rate diagram shown in Figure 2, which leads to the following equations:

\[
\frac{dP_{1,1}(t)}{dt} = -(\lambda + \alpha)P_{1,1}(t) + P_{1,0}(0,t), \quad (1)
\]

\[
\frac{\partial}{\partial t}P_{1,0}(u,t) = -\lambda P_{1,0}(u,t) + (\lambda c + \alpha c)P_{1,1}(u,t) + b(u)P_{0,0}(0,t) + \beta P_{1c}(u,t), \quad (2)
\]
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\[
\frac{dP_{IC}(t)}{dt} = -(\alpha + \beta)P_{IC}(t) + \lambda(1-c)P_{11}(t),
\]
\[
\frac{dP_{ID}(t)}{dt} = -\lambda P_{ID}(t) + \alpha(1-c)P_{11}(t),
\]
\[
\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\right)P_{00}(u,t) = \lambda P_{10}(u,t) + \alpha P_{IC}(u,t) + \lambda P_{ID}(u,t).
\]

In steady-state, we define
\[
P_{m,n} = \lim_{t \to \infty} P_{m,n}(t), \quad (m, n) = (1, 1), (1, 0), (0, 0), IC, ID,
\]
\[
P_{m,n}(u) = \lim_{t \to \infty} P_{m,n}(u,t), \quad (m, n) = (1, 1), (1, 0), (0, 0), IC, ID
\]
and further define
\[
P_{11}(u) = b(u)P_{11},
\]
\[
P_{IC}(u) = b(u)P_{IC},
\]
\[
P_{ID}(u) = b(u)P_{ID}.
\]

From (1)-(8), the following steady-state equations are given by
\[
0 = -(\lambda + \alpha)P_{11} + P_{10}(0),
\]
\[
-\frac{d}{du}P_{10}(u) = -\lambda P_{10}(u) + \left(\lambda c + \alpha c\right)b(u)P_{11} + b(u)P_{00}(0) + \beta b(u)P_{IC},
\]
\[
0 = -(\alpha + \beta)P_{IC} + \lambda(1-c)P_{11},
\]
\[
0 = -\lambda P_{ID} + \alpha(1-c)P_{11},
\]
\[
-\frac{d}{du}P_{00}(u) = \lambda P_{10}(u) + \alpha b(u)P_{IC} + \lambda b(u)P_{ID}.
\]

Now, from (9) and (11)-(12), we get
\[
P_{10}(0) = (\lambda + \alpha)P_{11},
\]
\[
P_{IC} = \frac{\lambda(1-c)}{\alpha + \beta}P_{11},
\]
\[
P_{1D} = \frac{\alpha(1-c)}{\lambda}P_{11}.
\]

Further define
\[
B'(s) = \int_0^s e^{-su} b(u)du,
\]
\[
P_{m,n}^*(s) = \int_0^s e^{-su} P_{m,n}(u)du,
\]
\[
P_{m,n} = \int_0^\infty P_{m,n}(u)du,
\]
and
\[
\int_0^\infty e^{-u} \frac{d}{du} P_{m,n}(u) \, du = s P_{m,n}^*(s) - P_{m,n}(0).
\]

Now, taking the LST on both sides of (10), (13) and using (14)-(16) yields

\[
(\lambda - s)P_{0,0}^*(s) = B'(s)P_{0,0}(0) + \left[ \lambda c + \alpha c_s + \frac{\lambda \beta(1-c)}{\alpha + \beta} \right] B'(s)P_{1,1} - (\lambda + \alpha)P_{1,1},
\]

\[
-sP_{0,0}^*(s) = \lambda P_{0,0}^*(s) + \left[ \frac{\lambda \alpha(1-c)}{\alpha + \beta} + \alpha(1-c_s) \right] B'(s)P_{1,1} - P_{0,0}(0).
\]

A recursive method is used to develop the explicit expressions \( P_{m,n}^*(0) \), where \((m, n) = (0,0), (1,0)\). Setting \( s = \lambda \) and \( s = 0 \) in (17), respectively, we obtain

\[
0 = B'(\lambda)P_{0,0}(0) + \left[ \lambda c + \alpha c_s + \frac{\lambda \beta(1-c)}{\alpha + \beta} \right] B'(\lambda)P_{1,1} - (\lambda + \alpha)P_{1,1},
\]

\[
\lambda P_{0,0}^*(0) = P_{0,0}(0) + \left[ \lambda c + \alpha c_s + \frac{\lambda \beta(1-c)}{\alpha + \beta} \right] P_{1,1} - (\lambda + \alpha)P_{1,1},
\]
or

\[
\lambda P_{1,1}^*(0) = \frac{(\lambda + \alpha) - (\lambda + \alpha)B'(\lambda)}{\lambda B'(\lambda)} P_{1,1} = \frac{(\lambda + \alpha)[1 - B'(\lambda)]}{\lambda B'(\lambda)} P_{1,1}.
\]

Differentiating (18) with respect to \( s \) and setting \( s = 0 \) in the result, we finally have

\[
P_{0,0}^*(0) = -\lambda P_{1,0}^*(0) + b_1 \left[ \frac{\lambda \alpha(1-c)}{\alpha + \beta} + \alpha(1-c_s) \right] P_{1,1},
\]

where \( b_1 = -B'(0) \) denotes the mean repair time.

Likewise, differentiating (17) with respect to \( s \) and setting \( s = 0 \) in the result yields

\[
\lambda P_{1,0}^*(0) = P_{0,0}^*(0) - b_1 P_{0,0}(0) - b_1 \left[ \lambda c + \alpha c_s + \frac{\lambda \beta(1-c)}{\alpha + \beta} \right] P_{1,1}.
\]

From (21) and (22), the steady-state solution \( P_{0,0}^*(0) \) is given by

\[
P_{0,0}^*(0) = -P_{1,0}^*(0) + b_1 P_{0,0}(0) + b_1 \left[ \frac{\lambda \alpha(1-c)}{\alpha + \beta} + \alpha(1-c_s) \right] P_{1,1}
\]

\[
+ b_1 \left[ \frac{\lambda \alpha(1-c)}{\alpha + \beta} + \alpha(1-c_s) \right] P_{1,1}
\]

\[
= \frac{(\lambda + \alpha)[1 - B'(\lambda)]}{\lambda B'(\lambda)} P_{1,1} + \frac{(\lambda + \alpha)b_1}{B'(\lambda)} P_{1,1} + b_1 \left[ \frac{\lambda \alpha(1-c)}{\alpha + \beta} + \alpha(1-c_s) \right] P_{1,1},
\]

\[
= \left\{ \frac{(\lambda + \alpha)[1 - B'(\lambda)]}{\lambda B'(\lambda)} + \frac{(\lambda + \alpha)b_1}{B'(\lambda)} + \frac{\lambda \alpha(1-c)b_1}{\alpha + \beta} + \alpha(1-c_s)b_1 \right\} P_{1,1}.
\]
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Substituting (15)-(16), (20) and (23) in the normalizing condition

\[ P_{1,1} + P_{1,0}^*(0) + P_{0,0}^*(0) + P_{1C} + P_{1D} = 1, \]

we obtain

\[ P_{1,1} = \frac{1}{1 + \frac{\lambda(1-c)(1+\alpha b_1)}{\alpha + \beta} + \frac{(\lambda + \alpha)b_1}{B'(\lambda)} + \alpha(1-c_s)(\frac{1}{\lambda} + b_1)}. \]  \hspace{1cm} (24)

Substituting (24) into (16) and (20), respectively, we get

\[ P_{1D} = \frac{\alpha(1-c_s)}{1 + \frac{\lambda(1-c)(1+\alpha b_1)}{\alpha + \beta} + \frac{(\lambda + \alpha)b_1}{B'(\lambda)} + \alpha(1-c_s)(\frac{1}{\lambda} + b_1)}. \]  \hspace{1cm} (25)

and

\[ P_{1,0}^*(0) = \frac{\lambda + \alpha}{\lambda B'(\lambda)} \frac{\lambda + \alpha}{1 + \frac{\lambda(1-c)(1+\alpha b_1)}{\alpha + \beta} + \frac{(\lambda + \alpha)b_1}{B'(\lambda)} + \alpha(1-c_s)(\frac{1}{\lambda} + b_1)}. \]  \hspace{1cm} (26)

Assume both states (0, 0) and IC are system down states; then the steady-state availability for the first system, $A_{V_1}$, is

\[ A_{V_1} = P_{1,1} + P_{1D} + P_{1,0}^*(0) \]

\[ = \frac{\lambda + \alpha}{\lambda B'(\lambda)} \frac{\alpha(1-c_s)}{1 + \frac{\lambda(1-c)(1+\alpha b_1)}{\alpha + \beta} + \frac{(\lambda + \alpha)b_1}{B'(\lambda)} + \alpha(1-c_s)(\frac{1}{\lambda} + b_1)}. \]  \hspace{1cm} (27)

4.1. Special cases

We consider five special cases for five different repair time distributions such as exponential (M), gamma (G), uniform (U), deterministic (D), and normal (N). The explicit expressions for the $A_{V_1}$ for five different repair time distributions such as exponential, gamma, uniform, deterministic, and normal are listed as follows.

Case 1. The repair time has an exponential distribution. We set the mean repair time $b_1 = 1/\mu$, where $\mu$ is the repair rate. In this case, we have

\[ B'(s) = \frac{\mu}{\mu + s}. \]

Using formula (27), we get the explicit expression for the $A_{V_1}(M)$:

\[ A_{V_1}(M) = \frac{1 + \frac{\lambda + \alpha}{\mu} + \frac{\alpha(1-c_s)}{\lambda}}{1 + \frac{\lambda(1-c)(1+\alpha b_1)}{\mu(1+\lambda)} + \frac{(\lambda + \alpha)b_1}{(1+\lambda)(\mu + \lambda)} + \frac{\alpha(1-c_s)(1/\lambda + 1/\mu)}{\mu}}. \]  \hspace{1cm} (28)
Case 2. The repair time has a gamma distribution with parameters \( \gamma \) and \( \mu \). We set the mean repair time \( b_i = 1/ \mu \), where \( \mu \) is the repair rate. In this case, we have

\[
B'(s) = \left( \frac{\gamma \mu}{\gamma \mu + s} \right). 
\]

It follows from formula (27) again, the explicit expression for the \( Av_1(G) \) is given by

\[
Av_1(G) = \frac{\lambda + \alpha}{\mu} (1 + \frac{\lambda}{r \mu}) \gamma - \frac{\alpha c_i}{\lambda} 
\]

\[\times 1 + \frac{\lambda + \alpha}{\mu} (1 + \frac{\lambda}{r \mu}) \gamma + \frac{\lambda(1-c)}{\alpha + \beta} (1 + \frac{\lambda}{r \mu}) + \alpha(1-c_i)(\frac{1}{\lambda} + \frac{1}{\mu})\].

Case 3. The repair time has a uniform distribution over the interval \([a, b]\). We set the mean repair time \( b_i = (a + b)/2 \). In this case, we have

\[
B'(s) = \frac{e^{-st} - e^{-st}}{s(a + b)}. 
\]

Using formula (27) above, we obtain the explicit expression for the \( Av_1(U) \):

\[
Av_1(U) = \frac{\lambda + \alpha}{\mu e^{-st} - e^{-st}} - \frac{\alpha c_i}{\lambda} 
\]

\[\times 1 + \frac{\lambda + \alpha}{\mu} (1 + \frac{\lambda}{r \mu}) \gamma + \frac{\lambda(1-c)}{\alpha + \beta} (1 + \frac{\lambda}{r \mu}) + \alpha(1-c_i)(\frac{1}{\lambda} + \frac{1}{\mu})\].

Case 4. The repair time distribution is deterministic. We set the mean repair time \( b_i = 1/ \mu \). In this case, we have

\[
B'(s) = e^{-st}. 
\]

From formula (27) again, the explicit expression for the \( Av_1(D) \) is given by

\[
Av_1(D) = \frac{\lambda + \alpha e^{\frac{\lambda}{\mu}}}{\lambda} - \frac{\alpha c_i}{\lambda} 
\]

\[\times 1 + \frac{\lambda + \alpha e^{\frac{\lambda}{\mu}}}{\mu} + \frac{\lambda(1-c)}{\alpha + \beta} (1 + \frac{\lambda}{r \mu}) + \alpha(1-c_i)(\frac{1}{\lambda} + \frac{1}{\mu})\].

Case 5. The repair time has a normal distribution with mean \( \mu \) and variance \( \sigma^2 \).

In this case, we have

\[
B'(s) = e^{-st + \frac{1}{2}s^2\sigma^2}, 
\]

and hence, using formula (27), we get the explicit expression for the \( Av_1(N) \)

\[
Av_1(N) = \frac{(\lambda + \alpha e^{\frac{(\lambda-0.5s^2\sigma^2)}{\mu}})}{\lambda} - \frac{\alpha c_i}{\lambda} 
\]

\[\times 1 + \frac{(\lambda + \alpha e^{\frac{(\lambda-0.5s^2\sigma^2)}{\mu}})}{\mu} + \frac{\lambda(1-c)}{\alpha + \beta} (1 + \frac{\lambda}{r \mu}) + \alpha(1-c_i)(\frac{1}{\lambda} + \frac{1}{\mu})\].

From Trivedi [13, p. 469], we have
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\[
\pi_{1,0} = \pi_{1,1} \frac{\lambda + \alpha}{\mu}, \\
\pi_{1D} = \pi_{1,1} \frac{\alpha(1-c_{i})}{\mu},
\]

\[
\pi_{1,1} = \frac{1}{1 + \frac{\lambda + \alpha}{\mu}(1 + \frac{\lambda}{\mu}) + \frac{\lambda(1-c_{i})}{\mu}(1 + \frac{\alpha}{\mu}) + \alpha(1-c_{i})(1 + \frac{1}{\lambda} + \frac{1}{\mu})}.
\]

It implies that the explicit expression for the \(Av(M)\) is given by

\[
Av_{i}(M) = \pi_{1,0} + \pi_{1D} + \pi_{1,1}
\]

\[
= \frac{1 + \frac{\lambda + \alpha}{\mu} + \frac{\alpha(1-c_{i})}{\lambda}}{1 + \frac{\lambda(1-c_{i})(1+\alpha h)}{\alpha + \beta} + \frac{(\lambda + \alpha)h_{i}}{B'(\lambda)} + \alpha(1-c_{i})(1 + \frac{1}{\lambda} + h_{i})},
\]

which is in accordance with the expression (28). Hence, we demonstrate that the first system generalizes the Trivedi's system.

5. Availability Analysis of the Second System

For the second system, we consider a three-unit system which contains one active unit and two warm standby units. The state-transition-rate diagram of the second system is shown in Figure 3. We describe the second system in the following. Initially three units are working and the system is in state (1,2). When the active unit fails, with probability \(c\) the protection switch successfully recovers repair by switching in the warm standby state, and the system enters state (1,1). When the active unit fails, with probability \(1-c\) the protection switch unsuccessfully recovers the breakdown of the active unit and the system goes in state 2C. The breakdown of the warm standby unit while the active unit is still working is detected instantly with probability \(c_{i}\), and when this occurs, the system goes in state (1,1). If the breakdown of the warm standby unit is not detected (with probability \(1-c_{i}\)), the system enters state 2D. In state (1,1) when the active unit (or the warm standby unit) breaks down, with probability \(1-c\) (or \(1-c_{i}\)), the protection switch fails to cover the breakdown of the active unit (or the warm standby unit) and the system goes in state 1C (or state 1D). In state 2D when the active unit (or the warm standby unit) breaks down, with probability \(1-c\) (or \(1-c_{i}\)), the protection switch unsuccessfully recovers the breakdown of the active unit (or the warm standby unit) and the system goes in state 1C (or state 1D). In state (1,1) or state 2D, when the active unit breaks down, with probability \(c\) the protection switch successfully recovers repair by switching in the warm standby state, and the system enters state (1,0). If a unit breaks down when the system is in one of the states: (1,0), 1C, or 1D, the system fails and goes in state (0,0).

We define

\[
P_{1,2}(u) = b(u) P_{1,2},
\]

\[
P_{2C}(u) = b(u) P_{2C},
\]

\[
P_{2D}(u) = b(u) P_{2D},
\]

\[
P_{1C}(u) = b(u) P_{1C}.
\]
Figure 3. State-transition-rate diagram for the second system.

Referring to Figure 3 and following the same procedures given in Section 4 that analyzes the first system, we obtain the following steady-state equations:

\[ 0 = -(\lambda + 2\alpha)P_{1,2} + P_{1,1}(0), \]  
\[ \frac{d}{du} P_{1,1}(u) = -(\lambda + \alpha)P_{1,1}(u) + (\lambda c + 2\alpha c) b(u) P_{1,2} + b(u) P_{1,0}(0) + \beta b(u) P_{2C}, \]  
\[ 0 = -(2\alpha + \beta)P_{1C} + \lambda(1-c)P_{1,2}, \]  
\[ 0 = -(\lambda + \alpha)P_{2D} + 2\alpha(1-c)P_{1,2}, \]

\[ \frac{d}{du} P_{1,0}(u) = -\lambda P_{1,0}(u) + (\lambda c + \alpha c) P_{1,1}(u) + b(u) P_{0,0}(0) + \beta b(u) P_{1C}, \]
\[ + (\lambda c + \alpha c) b(u) P_{2D}, \]
\[ 0 = -(\alpha + \beta)P_{1C} + 2\alpha P_{2C} + \lambda(1-c)P_{11} + \lambda(1-c)P_{2D}, \]
\[ 0 = -\lambda P_{1D} + \alpha(1-c)P_{2D} + \alpha(1-c)P_{11}, \]
\[ \frac{d}{du} P_{0,0}(u) = \lambda P_{1,0}(u) + \alpha b(u) P_{1C} + \beta b(u) P_{1D}. \]

Now, from (29), (31)-(32), and (34)-(35), we obtain

\[ P_{1,0}(0) = (\lambda + 2\alpha)P_{1,2}, \]
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\[ P_{2C} = \frac{\lambda (1-c)}{2\alpha + \beta} P_{1,2}, \quad (38) \]

\[ P_{2D} = \frac{2\alpha (1-c)}{\lambda + \alpha} P_{1,2}, \quad (39) \]

\[ P_{IC} = \frac{2\alpha}{\alpha + \beta} P_{2C} + \frac{\lambda (1-c)}{\alpha + \beta} P_{1,1} + \frac{\lambda (1-c)}{\alpha + \beta} P_{2D}, \quad (40) \]

\[ P_{1D} = \frac{\alpha (1-c)}{\lambda} P_{2D} + \frac{\alpha (1-c)}{\lambda} P_{1,1}. \quad (41) \]

Taking the LST on both sides of (30), (33), (36) and using (37)-(41) yields

\[ (\lambda + \alpha + s)P_{1,1}^*(s) = B^*(s) P_{1,0}(0) + \left[ \frac{\lambda c + \alpha c_s + \lambda \beta (1-c)}{2\alpha + \beta} \right] B^*(s) P_{1,2} - (\lambda + 2\alpha)P_{1,2}, \quad (42) \]

\[ (\lambda - s)P_{1,0}^*(s) = B^*(s) P_{0,0}(0) + (\lambda c + \alpha c_s)P_{1,1}^*(s) + \beta B^*(s) P_{IC} \]

\[ + \frac{2\alpha (\lambda c + \alpha c_s)(1-c_s)}{\alpha + \alpha} B^*(s) P_{1,2} - P_{1,0}(0), \quad (43) \]

\[ -sP_{0,0}^*(s) = \lambda P_{0,0}^*(s) + \alpha B^*(s) P_{IC} + \lambda B^*(s) P_{1D} - P_{0,0}(0). \quad (44) \]

We develop a recursive method to get the explicit expression \( P_{m,n}^*(0) \), where \((m, n) = (0,0), (1,0)\). Setting \( s = \lambda + \alpha, \ s = 0 \) and \( s = \lambda \) in (42) yields

\[ P_{1,0}(0) = \frac{B^*(s) (\lambda + \alpha)}{\lambda + \alpha} B^*(s) P_{1,2}, \quad (45) \]

\[ (\lambda + \alpha)P_{1,1}^*(0) = \left[ \frac{\lambda + 2\alpha}{B^*(s)(\lambda + \alpha)} - (\lambda + 2\alpha) \right] P_{1,2}, \quad (46) \]

\[ P_{1,1}^*(\lambda) = \frac{(\lambda + 2\alpha)B^*(\lambda) - B^*(\lambda + \alpha)}{\alpha B^*(\lambda + \alpha)} P_{1,2}. \quad (47) \]

Setting \( s = \lambda \) and \( s = 0 \) in (43) and using (47), we finally obtain

\[ P_{1,0}(0) = B^*(\lambda) P_{0,0}(0) + (\lambda c + \alpha c_s)P_{1,0}^*(\lambda) + \beta B^*(\lambda) P_{IC} \]

\[ + \frac{2\alpha (\lambda c + \alpha c_s)(1-c_s)}{\alpha + \alpha} P_{1,2} - P_{1,0}(0) \]

\[ = B^*(\lambda) P_{0,0}(0) + \left\{ \frac{(\lambda c + \alpha c_s)(\lambda + 2\alpha)B^*(\lambda + \alpha)}{\alpha B^*(\lambda + \alpha)} \right\} P_{1,2} + \beta B^*(\lambda) P_{IC} - P_{1,0}(0), \quad (48) \]

\[ \lambda P_{1,0}^*(0) = P_{0,0}(0) + (\lambda c + \alpha c_s)P_{1,1}^*(0) + \beta P_{IC} + \frac{2\alpha (\lambda c + \alpha c_s)(1-c_s)}{\lambda + \alpha} P_{1,2} - P_{1,0}(0). \quad (49) \]

Again, setting \( s = 0 \) in (44) yields

\[ \lambda P_{0,0}^*(0) = P_{0,0}(0) + (\lambda c + \alpha c_s)P_{1,1}^*(0) + \beta P_{IC} + \frac{2\alpha (\lambda c + \alpha c_s)(1-c_s)}{\lambda + \alpha} P_{1,2} - P_{1,0}(0). \]
Differentiating (42)-(44) with respect to \( s \) and setting \( s = 0 \), we finally obtain

\[
0 = \lambda P_{1,0}^s(0) + \alpha P_{1c} + \lambda P_{1D} - P_{0,0}(0).
\]  

We sum up (51)-(53) yielding

\[
0 = -b_1 P_{0,0}(0) + \frac{-b_1 (\lambda c + \alpha c_s)}{\lambda + \alpha} P_{1,0}(0) - b_1 \frac{\lambda c + 2\alpha c_s + \lambda \beta (1-c)}{2\alpha + \beta} P_{1,2},
\]

where \( b_1 = -B_{s}^{(1)}(0) \) is the mean repair time. The corresponding normalizing condition is given by

\[
P_{1,2} + P_{2c} + P_{1,0}^s(0) + P_{1D} + P_{1c}^s(0) + P_{1D} + P_{0,0}^s(0) = 1.
\]
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\[
\begin{align*}
\begin{pmatrix}
-2\lambda + 2\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\lambda (1-c) & -(2\alpha + \beta) & 0 & 0 & 0 & 0 & 0 & 0 \\
2\alpha(1-c) & 0 & 0 & -\lambda & 0 & 0 & 0 & 0 \\
0 & 2\alpha & -\lambda \alpha + \beta & \lambda (1-c) & 0 & \lambda (1-c) & 0 & 0 \\
0 & 0 & 0 & \alpha(1-c) & -\lambda & \alpha(1-c) & 0 & 0 \\
(\lambda + 2\alpha)(1 - b_1) & 0 & 0 & 0 & 0 & -(\lambda + \alpha) & 0 & 0 \\
\theta_1 (b_2 - b_1) & 0 & \beta b_2 & 0 & 0 & 0 & 0 & 0 \\
\theta_2 & 0 & \beta & 0 & 0 & \lambda c + \lambda c & -\lambda & 0 \\
0 & 0 & \alpha & 0 & \alpha & 0 & \alpha & 0 \\
-\theta_3 b_1 & 0 & -\lambda b_1 & 0 & -\lambda b_1 & \frac{\lambda c + \lambda c}{\lambda c + \alpha} & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\end{align*}
\]

where

\[
b_2 = B'(\lambda), \quad b_3 = B'(\lambda + \alpha), \quad \theta_1 = \frac{(\lambda + 2\alpha)(\lambda c + \lambda c)}{\alpha},
\]

\[
\theta_2 = \frac{2\alpha(\lambda c + \lambda c)(1 - c_1)}{\lambda c + \lambda c}, \quad \theta_3 = \frac{(\lambda c + \lambda c)[(\lambda c + \lambda c + 2\alpha + \frac{\lambda c + \lambda c}{\lambda c + \lambda c}]}{2\alpha + \beta}.
\]

Solving (56), we can obtain the steady-state probabilities \( P_{0,0} \) and \( P_{1,c} \). Assume both states (0, 0) and JC are system down states; then the steady-state availability for the second system, \( A_{V_2} \), is given by

\[
A_{V_2} = 1 - (P_{0,0} + P_{1,c}). \tag{57}
\]

5.1. Special cases

For the second system, substituting the expressions \( B'(s) \) for five different repair time distributions: exponential (M), gamma (G), uniform (U), deterministic (D), and normal (N) in (56), we can obtain the \( A_{V_2} \) for five different repair time distributions: exponential, gamma, uniform, deterministic, and normal, respectively. The steady-state availability \( A_{V_2} \) are too ample to be shown here. However, based on assumed numerical values, we use an efficient Maple computer program to calculate the corresponding \( A_{V_2} \) for five different repair time distributions.

6. Comparison of the Two Systems

In this section, two availability systems will be compared in terms of their \( A_{V_i} \) \( (i = 1, 2) \) for five different repair time distribution: exponential, gamma, uniform, deterministic and normal. Basically, we consider the following three cases:

Case 1. \( \alpha = 0.2 \lambda, \mu = 1.0, \beta = 2.4, c = 0.9, c_i = 0.8, \) vary the values of \( \lambda \) from 0.1 to 1.0.

Case 2. \( \lambda = 0.3, \alpha = 0.2 \lambda, \beta = 2.4, c = 0.9, c_i = 0.8, \) vary the values of \( \mu \) from 0.6 to 1.5.

Case 3. \( \lambda = 0.3, \alpha = 0.2 \lambda, \mu = 1.0, c = 0.9, c_i = 0.8, \) vary the values of \( \beta \) from 1 to 10.
6.1. Comparison of the first system

We perform a comparison for the \( Av_i \) when the repair time distribution is exponential, gamma, uniform, deterministic and normal. Numerical results of the \( Av_i(M) \), \( Av_i(G) \), \( Av_i(U) \), \( Av_i(D) \) and \( Av_i(N) \) for the first system are displayed in Figures 4-6 for cases 1-3, respectively. From Figure 4 and Figure 6, we observe that (i) \( Av_i(D) > Av_i(U) > Av_i(G) > Av_i(M) > Av_i(N) \) for cases 1 and 3; (ii) \( Av_i \) decreases as \( \lambda \) increases; and (iii) \( Av_i \) increases as \( \beta \) increases. Figure 5 reveals that (i) when \( 0.6 < \mu < 0.9071 \), \( Av_i(D) > Av_i(U) > Av_i(G) > Av_i(N) > Av_i(M) \); (ii) when \( 0.9071 < \mu < 1.6 \), \( Av_i(D) > Av_i(U) > Av_i(G) > Av_i(M) > Av_i(N) \); and (iii) \( Av_i \) increases as \( \mu \) increases.

Figure 4. Steady-state availability versus \( \lambda \).

Figure 5. Steady-state availability versus \( \mu \).
6.2. Comparison of the second system

For the second system, we perform a comparison for the $Av_2$ when the repair time distribution is exponential, gamma, uniform, deterministic and normal. Numerical results of the $Av_2(M)$, $Av_2(G)$, $Av_2(U)$, $Av_2(D)$ and $Av_2(N)$ are shown in Figures 7-9 for cases 1-3, respectively. It appears from Figure 7 that (i) when $0.1 < \lambda < 0.5295$, $Av_2(D) > Av_2(U) > Av_2(G) > Av_2(N) > Av_2(M)$; (ii) when $0.5295 < \lambda < 1.0$, $Av_2(D) > Av_2(U) > Av_2(G) > Av_2(M) > Av_2(N)$; and (iii) $Av_1$ decreases as $\lambda$ increases. One sees from Figure 8 that (i) when $0.6 < \mu < 0.6385$, $Av_2(D) > Av_2(U) > Av_2(N) > Av_2(G) > Av_2(M)$; (ii) when $0.6385 < \mu < 1.0828$, $Av_2(D) > Av_2(U) > Av_2(G) > Av_2(N) > Av_2(M)$; (iii) when $1.0828 < \mu < 1.6$, $Av_2(D) > Av_2(U) > Av_2(G) > Av_2(M) > Av_2(N)$; (iv) $Av_1$ increases as $\mu$ increases. It can be easily see from Figure 9 that (i) $Av_2(D) > Av_2(U) > Av_2(G) > Av_2(N) > Av_2(M)$; and (ii) $Av_2$ increases as $\beta$ increases.
7. Conclusion

In this paper, we have used the supplementary variable technique to derive the steady-state availability for two systems. We have shown that the first system generalizes the Trivedi's system (Markovian system). Next, we perform the comparison between five different repair time distributions in two systems and we show the constant distribution (deterministic) is better than the other distributions in two systems. Finally, from the comparison of two systems we can draw the conclusion that, the second system is better than the first system when increasing the number of warm standby units.
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References


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