The Economic Performance of a CUSUM $t$ Control Chart for Monitoring Short Production Runs

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Abstract: At the start of a production run, the parameters of the distribution of a quality characteristic are frequently unknown. If the quality characteristic should be monitored on-line by means of a control chart and the production run length is too short, then it is impossible to get a sufficient Phase I number of samples allowing for the correct estimation of parameters and the evaluation of the chart's tentative limits. To overcome this problem, in this paper the economic design of a CUSUM $t$ control chart that is able to monitor short production runs without the need of Phase I samples is proposed. A numerical analysis aimed at comparing the economic performance of the CUSUM $t$ chart vs. the CUSUM $\bar{X}$ chart with known parameters has been performed on a benchmark of scenarios available from literature. The obtained results show that the expected economic loss associated to the implementation of the CUSUM $t$ chart is quite negligible with respect to the control chart with known parameters. An illustrative example is provided to show the CUSUM $t$ chart’s implementation.

Keywords: Costs, CUSUM control chart, quality control, short production run, unknown process parameters.

1. Introduction

A high degree of flexibility and the potential of producing a large portfolio of items is a key factor for companies to successfully operate in the current worldwide market characterized by aggressive pricing competitiveness. Manufacturing small production runs of high standard and customized products is vital for business of small enterprises which cannot grasp the benefits of the economies of scale and are not able to achieve so significant cost cutting as competing companies located in the emerging markets. Job shop manufacturing of limited series of mechanical parts by means of multi-purpose Computer Numerical Control (CNC) centers represents a typical example of production of limited lots of parts in a Just-in-Time environment. Other examples of small run productions can be found in the clothes / fashion industry and the production of customized furniture products.

Implementing control charts in short production runs is a challenging issue and thus has received careful attention in quality control literature. Frequently, productions with short rolling horizons are limited to a few hours to manufacture small lots of products. Consequently, performing SPC activities should be restricted to a few number of inspections. In the absence of preliminary process knowledge, the quality practitioner cannot gather enough information to get correct estimates about the in-control population mean $\mu_0$ and standard deviation $\sigma_0$ of the quality characteristic.

Quesenberry [14] demonstrated that at least $\hat{I}_S = 400/(n-1)$ Phase I samples should be
taken to get estimated limits close to the true chart’s limits. To reduce the number \( \hat{I}_S \) of preliminary subgroups needed for the estimation of population parameters, Neduraman and Pigniatelli [11] and Tsai et al. [17] have proposed different procedures based on evaluating statistics having a \( t \)-distribution which allow a set of prospective control limits to be constructed time by time for a specific number \( k \) of future subgroups. These approaches still need the availability of a number \( \hat{I}_S \) of initial subgroups; thus, they are more effective in medium to large-run production manufacturing scenarios to implement self-starting control charts than in not repetitive short-runs of small batches of parts, where the number of inspections can be actually smaller than \( \hat{I}_S \).

Self-starting control charts have been proposed as a means to monitor start-up processes or short runs. Quesenberry [13] has defined a set of sequential \( Q \) statistics to be used in Shewhart, EWMA and CUSUM charts for the detection of changes in the process mean or variance. Del Castillo and Montgomery [5] and Zantek [19] have investigated the statistical properties of the \( Q \) charts and proposed some enhancements. He et al. [7] have demonstrated that the \( Q \) charts with unknown variance are biased; that is, for some specific mean shift sizes the \( Q \) charts have larger out-of-control average run lengths than the expected number of false alarms. This problem, known as “masking of the shift”, is particularly obvious if the shift occurs at the early stage of sampling. The modified \( Q \) control charts proposed by He et al. [7] still present some bias and difficulty of implementation in practice.

More recently, Zhang et al. [20] have compared the statistical properties of Shewhart and EWMA \( t \) and \( \bar{X} \) charts when the in-control process standard deviation is not correctly estimated. They have proven that the \( t \) charts are more robust than the \( \bar{X} \) charts to a misspecification of the in-control process standard deviation estimation. The implementation of Shewhart and EWMA \( t \) control charts has been successfully extended to short production runs by Celano et al. [4]: the main motivation leading to implementing the \( t \) charts in short production runs is that they do not require any preliminary estimation of the process parameters and allow for the SPC to be immediately started with the production run, without recurring to any \( \hat{I}_S \) preliminary inspections. In Celano et al. [4] the statistical properties of the \( t \) charts are compared with those of the Shewhart and EWMA charts with known process parameters under both the assumptions of perfect and imperfect initial setup.

To design a control chart, the following parameters should be selected: the sample size \( n \), the sampling interval \( h \) and the width of control limits \( k \). From a practical point of view, choosing the chart’s design parameters which minimize the expected quality control cost during the run is an appealing management strategy.

In short production runs, the pioneering economic model for the design of control charts for monitoring the fraction non-conforming has been proposed by Ladany [9]. A dynamic control chart modelled as a generalization of an EWMA scheme has been investigated by Wasserman [18]. Del Castillo and Montgomery [6] have proposed a general economic model to design control charts for both the short and long runs allowing for the optimization of the sample size, the width of the control limits and the number of samples. Recently, Ho and Trindade [8] proposed the economic design of an individual \( X \) chart for short runs with additional retrospective inspection. The economic design and the performance properties of a CUSUM \( \bar{X} \) chart for short production runs with known parameters were investigated in Nenes and Tagaras [12]. Trovato et al. [16] have proposed a comparison of the economic performance of several control charts for monitoring the sample variance in short-runs.

All of these papers dealing with the economic design of control charts for short production runs consider as known the in-control process parameters, i.e. the in-control
population mean $\mu_0$ and the standard deviation $\sigma_0$ of the quality characteristic. Often, this simplifying assumption does not hold in practice and, in particular, within flexible manufacturing systems producing non-repetitive runs; thus, it needs to be removed because of the lack of a sufficient number of data to perform the estimation of the parameters. Furthermore, due to the limited time horizon of the short-run and the frequent absence of properly designed inspection devices to monitor each specific part code, the time length needed to produce and then inspect a part code should be considered during the chart’s design to avoid the selection of unfeasible design parameters for the chart itself.

This paper investigates the economic design of a SPC inspection procedure based on a CUSUM $t$ control chart for monitoring the mean during a short production run in the presence of finite rates of production and inspection. Implementing the $t$ statistic allows for removing the preliminary estimations of the in-control population mean $\mu_0$ and the standard deviation $\sigma_0$: thus, it is particularly suited for productions with few scheduled inspections. Of course, it results in an economic loss due to the lower sensitivity of the $t$-charts in the detection of anomalies than the “ideal” control chart with perfectly known parameters. To check if this economic loss can be maintained at an acceptable level and in order to propose the CUSUM $t$ chart as an easy to be handled SPC tool in short production runs, a direct comparison is performed with the CUSUM $\mathcal{R}$ chart with known parameters for monitoring a short-run, see Nenes and Tagaras [12].

The remainder of the paper is organized as follows: Section 2 discusses the configuration of the inspection activities and the related constraints. Section 3 presents a description of the economic model; Section 4 shows the results of the numerical analysis. An illustrative example is presented in Section 5 to show the implementation of the CUSUM $t$ chart. Conclusions and future research directions complete the paper. An appendix reports the formulas relative to the stochastic model adopted to cope with the investigated problem.

2. The Configuration of the Inspection Activities

A manufacturing process is scheduled to produce a finite population of $N$ parts during a finite production horizon having length $H$. Parts can be worked and then released individually or in small groups having size $B$ from the manufacturing to the local inspection area: as an example, this scenario can happen in the manufacturing of mechanical parts which are moved and worked together in the CNC machining centers by means of pallets and/or properly designed fixture systems. During the production run, the mean of a normally distributed quality characteristic $X \sim N(\mu, \sigma)$ having target value equal to $U$ should be monitored at a stage of the process by means of a control chart. The shift size of the mean is measured as a multiple $\delta$ of the in-control population standard deviation $\sigma_0$. Initial setup activities are performed on the process in order to start the production with the in-control population mean $\mu_0$ as close as possible to the target $U$. Let $I_s$ be the number of scheduled inspections within the production horizon $H$. No inspection is scheduled at the end of the run. Therefore, the interval between two consecutive inspections is $h = H/(I_s+1)$ hours.

Usually, the design of a control chart for a short run has been investigated without doing any reference to the batch dimension, the process productivity and the inspection equipment performance: in particular, the rate of production $r_{PR}$ [parts/hour] and the rate of inspection $r_i$ [parts/hour] at the process stage are implicitly assumed as infinite. Under this simplifying assumption, the sample size $n$ and the number $I_s$ of inspections (or, equivalently, the inspection interval $h$) are limited by the condition $n \cdot I_s \leq N$, where the equality between the left and the right member means the 100% sampling inspection. However, the rate of production $r_{PR}$ and the rate of inspection $r_i$ are finite and must be taken into account by the
quality practitioner during the chart design activities, because they can constrain the chart design selection.

We assume that the rational subgroup of the quality characteristic's measurements made available at the $i$th inspection time to plot the point on the chart should be collected among the parts produced within the previous time interval $[(i-1)h; ih]$. Thus, to avoid delays at the inspection area and to maintain the fixed inspection interval $h$, the sample size $n$ should be limited superiorly by:

$$n_{\text{max}} = \min \left( \left\lfloor \frac{H \cdot r_{\text{PR}}}{I_s + 1} \right\rfloor ; r_I \cdot \left[ \frac{H}{I_s + 1} \cdot \max \left( 0; B \cdot \left( \frac{1}{r_{\text{PR}}} - \frac{1}{r_I} \right) \cdot \left[ \frac{H \cdot r_{\text{PR}}}{I_s + 1} - 1 \right] \right) \right] \right)$$

(1)

where $\lfloor x \rfloor$ denotes the integer less than or equal to $x$. To illustrate the influence of the production and inspection rates on the rational subgroup size and to help readers in understanding the meaning of the terms in equation (1), the following example is presented. Figure 1 shows the schedule of the production and the inspection activities between the start of the run and the first inspection time in a single process stage where $N = H r_{\text{PR}} = 80$ parts are produced within a production horizon $H = 8$ hours. The rate of production of the short run is $r_{\text{PR}} = N/H = 10$ parts/hour. Without loss of generality, the number of scheduled inspections is $I_s = 4$. The first inspection occurs after $h = H/(I_s+1) = 1.6$ hours since the beginning of the short run, whereas the fourth (last) inspection is scheduled $h = 1.6$ hours before the end of the production horizon. Figures 1a) and 1b) show the condition $r_{\text{PR}} < r_I = 20$ parts/hour, when the batch size $B = 1, 2$. Conversely, Figures 1c) and 1d) show the condition $r_{\text{PR}} > r_I = 5$ parts/hour, when $B = 1, 2$.

FIGURES

![Diagram showing the schedule of the production and inspection activities](image)

Figure 1. Schedule of the production / inspection activities: influence on the maximum allowable sample size $n_{\text{max}}$. 
A numbered rectangle represents a worked, (or inspected), part. Parts belonging to the same batch have the same rectangle background colour. Between two successive sampling epochs \[ \frac{H}{(I+1)} \cdot r_{PR} \] = 16 parts are worked. The production activities allow for a maximum sample size \[ \frac{H}{(I+1)} \cdot r_{PR} - 1 \] = 15. Depending on the rate of inspection \( r_I \) and the batch \( B \) dimension, this maximum allowable sample size should be reduced further or not, accordingly to eqn. (1). For example, if \( r_{PR} < r_I = 20 \) parts/hour and \( B=2 \), see the Figure 1b), then:

\[
\begin{align*}
  n_{max} &= \min \left[ \frac{8 \cdot 10}{4 + 1} - 1 \right]; 20 \cdot \left[ \frac{8}{4 + 1} - \frac{2}{10} - \max \left( 0; 2 \cdot \left( \frac{1}{10} - \frac{1}{20} \right) \right) \left[ \frac{8 \cdot 10}{(4 + 1) \cdot 2} - 1 \right] \right] = 14. \quad (2)
\end{align*}
\]

Similarly, if \( r_{PR} > r_I = 5 \) parts/hour and \( B=2 \), see the Figure 1d), it results in:

\[
\begin{align*}
  n_{max} &= \min \left[ \frac{8 \cdot 10}{4 + 1} - 1 \right]; 5 \cdot \left[ \frac{8}{4 + 1} - \frac{2}{10} - \max \left( 0; 2 \cdot \left( \frac{1}{10} - \frac{1}{5} \right) \right) \left[ \frac{8 \cdot 10}{(4 + 1) \cdot 2} - 1 \right] \right] = 7. \quad (3)
\end{align*}
\]

In this case, the value of \( n_{max} \) is halved with respect to the \( n > r_{PR} \) condition. It appears immediately evident that the batch size \( B \) has a significant effect in reducing the maximum allowable sample size when \( n > r_{PR} \), i.e. when the inspection activities are blocked by the availability of worked parts released by the upstream production area, see the Figure 1b). Conversely, when the condition \( n < r_{PR} \) occurs, the rate of inspection \( r_I \) strongly constraints the value of \( n_{max} \); the batch size \( B \) can further reduce \( n_{max} \), see Figures 1c) and 1d). When the optimal sample size is \( n < n_{max} \), an “as late as possible” (ALAP) scheduling of the \( n \) measures should be decided by the quality practitioner to allow for an estimation of the process position as close as possible to the inspection epoch when the point is plotted on the chart.

Once a subgroup \( \{ X_{i,1}, X_{i,2}, \ldots, X_{i,n} \} \), \( i = 1, 2, \ldots, I_s \), of \( n > 1 \) measures related to the quality characteristic to be monitored is available at the generic \( i \)th inspection epoch, the subgroup mean \( \bar{X}_i \) and standard deviation \( S_i \) are immediately computed as follows:

\[
\begin{align*}
  \bar{X}_i &= \frac{1}{n} \sum_{j=1}^{n} X_{i,j}, \quad S_i = \sqrt{\frac{1}{n-1} \sum_{j=1}^{n} (X_{i,j} - \bar{X}_i)^2} \quad (4)
\end{align*}
\]

and the corresponding statistic \( T_i \) is defined as:

\[
\begin{align*}
  T_i &= \frac{\sqrt{n} \left( \bar{X}_i - U \right)}{S_i}. \quad (5)
\end{align*}
\]

Once the \( X_{i,j} \) measures are available, the time needed to get \( \bar{X}_i, S_i \) and \( T_i \) can be reasonably considered as negligible thanks to the huge availability of computer software automating mathematical and statistical computations.

Define the following notations for the \( T_i \) distribution functions:

\[
\begin{align*}
  F_{\ell} (|n-1|) & \text{ is the Student } t \text{ cumulative distribution function with } n-1 \text{ degrees of freedom; } \\
  G_{\ell} (|n-1|, \Psi) & \text{ is the non central Student } t \text{ cumulative distribution function with } n-1 \text{ degrees of freedom and non centrality parameter } \Psi.
\end{align*}
\]

Here, an initial perfect setup is assumed, i.e. at the beginning of the production, the in-control mean takes on the target value, i.e. \( \mu_0 = U \). Let us denote as \( Y_i \) the state of the process at the epoch \( i \): if the process is in-control, then \( Y_i = 0 \); conversely, if the process has
shifted to the out-of-control condition, then \( Y_i = 1 \). Thus, when the process operates in the in-control state (\( Y_i = 0 \)), the statistic \( T_i \sim F_i \left( \left| n-1 \right| \right) \). Once a special cause occurs (\( Y_i = 1 \)), the mean shifts from \( \mu_0 \) to \( \mu_1 = \mu_0 + \delta \sigma_0 \). By adding and subtracting the same quantity \( \mu_i \sqrt{n} \), the \( T_i \) statistic can be written as follows:

\[
T_i = \frac{\sqrt{n}(\bar{X} - U)}{S_i} = \frac{\sqrt{n}(\bar{X} - U) + \mu_1 \sqrt{n} - \mu_0 \sqrt{n}}{S_i}. \tag{6a}
\]

By rearranging the terms at the numerator of equation (6a) and dividing both the numerator and the denominator of equation (6a) by \( \sigma_0 \), it results in:

\[
T_i = \frac{\sqrt{n}(\bar{X} - \mu_1) + \mu_1 \sqrt{n} - U \sqrt{n}}{S_i} = \frac{\sqrt{n}(\bar{X} - \mu_1) / \sigma_0 + \sqrt{n}(\mu_1 - U) / \sigma_0}{S_i / \sigma_0}. \tag{6b}
\]

Finally, by replacing in the second term of the numerator \( \mu_1 \) with \( \mu_0 + \delta \sigma_0 \) and remembering that under the assumption of a perfect setup \( \mu_0 = U \) holds, after some calculations the following results are obtained:

\[
T_i = \frac{\sqrt{n}(\bar{X} - \mu_1) / \sigma_0 + \delta \sqrt{n} \sigma_0 / \sigma_0}{S_i / \sigma_0} = Z \frac{\sqrt{V}}{\sqrt{n-1}} \tag{6c}
\]

where \( Z = \frac{\sqrt{n}(\bar{X} - \mu_1) / \sigma_0} \) is a standard normal variable \( N(0, 1) \) and

\[
V = (n-1) \frac{S_i^2}{\sigma_0^2}
\]

is chi-squared \( \chi^2_{n-1} \) distributed. Thus, \( T_i \) has a non central \( t \) distribution with non centrality parameter \( \Psi = \delta \sqrt{n} \), i.e. \( T_i \sim G_i \left( \left| n-1, \delta \sqrt{n} \right| \right) \).

During the short run, the \( T_i \) statistic is monitored by means of a one-sided CUSUM \( t \)-control chart for detecting upward shifts in the process mean. An analogous formulation can be provided for downward shifts. The monitored CUSUM statistic is:

\[
C_i = \max \left\{ 0, C_i-1 + T_i - E(T) - k \right\}, \quad i = 1, \ldots, I_s, \quad E(T) = 0, \quad C_0 = 0 \quad \tag{7}
\]

where \( E(T) = 0 \) is the mean of the Student \( t \) distribution function with \( n-1 \) degrees of freedom. The production run starts in-control and perfectly on the target, therefore \( C_0 = 0 \). To start process monitoring, the number of inspections \( I_s \), the sample size \( n \), the control limit \( UCL \) and the reference value \( k \) of the CUSUM chart should be selected by achieving the economic objective of minimizing the quality control cost subject to the constraints related to the inspection and production rates.

### 3. The Economic Model

The expected quality control cost \( E(TC) \) associated to the implementation of a control chart is equal to the sum of the following terms: the out-of-control production cost; the cost of false alarms; the cost for searching and eventually eliminating a special cause; the sampling and inspection costs. Table 1 shows the notations adopted to define the cost model.

During the short run, the event of an assignable cause shifting the process state from \( Y_i = 0 \) to \( Y_i = 1 \) has a Poisson distribution with an exponentially distributed inter-arrival time...
having mean $1/\nu$, where $\nu$ [failures/hour] is the failure rate. Assuming that the process is in-control at the inspection epoch $i-1$, the probability $\gamma$ that an assignable cause will occur between two successive inspections $i-1$ and $i$ is equal to:

$$\gamma = 1 - e^{-\nu h}.$$  \hfill (8)

Given the finite length $H$ of the rolling horizon, the assignable cause may not occur at all during a production run. On the other hand, if an assignable cause occurs and is signalled, then the process will be perfectly restored to its nominal condition and will be restarted in order to complete the residual operating time. Of course, further failures can occur by the end of the run. When the process shifts from the in-control state $Y_i = 0$ to the out-of-control state $Y_i = 1$, operating in the out-of-control state is expensive in terms of both a lower output quality and higher rework costs. Actually, the expected number of nonconformities increases, thus involving a loss rate $M$ [$$/hour].

Table 1. Notations.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>sample size</td>
</tr>
<tr>
<td>$UCL$</td>
<td>CUSUM control limit</td>
</tr>
<tr>
<td>$k$</td>
<td>reference value</td>
</tr>
<tr>
<td>$h$</td>
<td>inspection interval</td>
</tr>
<tr>
<td>$r_{PR}$</td>
<td>rate of production</td>
</tr>
<tr>
<td>$r_i$</td>
<td>rate of inspection</td>
</tr>
<tr>
<td>$\alpha_0$</td>
<td>probability of a Type I error</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>chart's power</td>
</tr>
<tr>
<td>$U$</td>
<td>Quality characteristic target value</td>
</tr>
<tr>
<td>$\mu_0$</td>
<td>in-control population mean</td>
</tr>
</tbody>
</table>

$E(TC)$ can be evaluated by using an approach based on a two-dimensional Markov-chain approximation, see Nenes and Tagaras [12]. This procedure is described in the Appendix and adapts the Brook and Evans [1] and Lucas and Saccucci [10] approaches to the investigated process operating conditions. Accordingly to the economic model discussed in Nenes and Tagaras [12], the expected quality control cost $E(TC)$ is equal to:

$$E(TC) = (a + b \cdot n) I_e + M \left[ h - \frac{\nu}{\nu} \right] \left( 1 + \sum_{i=1}^{I_e} \sum_{j=0}^{m-1} p_{0,j}^{(i)} + \sum_{i=1}^{I_e} q_{0,m}^{(i)} \right) + M \cdot h \sum_{i=1}^{I_e} \sum_{j=0}^{m-1} q_{0,j}^{(i)} + L_0 \sum_{i=1}^{I_e} p_{0,m}^{(i)} + L_1 \sum_{i=1}^{I_e} q_{0,m}^{(i)}.$$  \hfill (9)

In the right hand side of equation (9), the first term represents the sampling cost; the second term is the expected out-of-control production cost when a shift occurs within the $[i \cdot h; (i+1) \cdot h]$ time interval; the third term is equal to the expected out-of-control production cost when the process is already out-of-control at the beginning of the $[i \cdot h; (i+1) \cdot h]$ time interval and there remains due a Type II error at the $i$th inspection epoch; the fourth term quantifies the expected false alarm cost; finally, the fifth term is the expected cost of eliminating a special cause. $p_{u,v}$ and $q_{u,v}$ ($u, v = 0, \ldots, m$) are the transition probabilities between the states $u$ and $v$ of the two-dimensional Markov chain with the process operating in the in-control state and shifted to the out-of-control state, respectively; $m$ is the number of transient states. More details about the stochastic model can be found in the Appendix.
Let us denote as \( \Omega = \{ I_s, n, k, UCL \} \) the generic design solution of the inspection procedure. Then, the mathematical constrained optimization problem related to the selection of the economic design for the one-sided CUSUM \( t \) chart or, alternatively, one among the “no control” strategy or the “preventive maintenance” without sampling strategy can be stated as follows:

\[
\min_{\Omega} \left[ E(TC) \right] \tag{10}
\]

subject to:

\[
0 \leq I_s \leq N/B \tag{11}
\]

\[
2 < n \leq n_{\text{max}} \tag{12}
\]

where constraint (11) accounts for the maximum allowable number of inspections to be scheduled during the rolling horizon and is a function of the number of parts \( N=H.r_{PR} \) to be produced and the dimension of batches grouping parts at the workstation; constraint (12) is related to the production and inspection rates and is fixed through equation (1).

The ranges of variation for the reference value \( k \) and the control limit \( UCL \) of the CUSUM chart are defined as follows: \( k \in \{0.1, 0.2, \ldots, 2\} \) and \( UCL \in \{0.05, 0.1, \ldots, 5\} \).

An exhaustive optimization algorithm has been properly designed to select the economic design of the on-line control strategy. The optimization procedure starts by setting the generic set of design parameters equal to \( \Omega = \emptyset \) and computing the cost \( TC_{NC} \) associated to a strategy calling for no-control during the short run:

\[
TC_{NC} = E(TC|I_s = 0) = M \left( H - \frac{1-e^{-h/k}}{\nu} \right). \tag{13}
\]

Then, the algorithm iteratively increases the number of scheduled inspections \( I_s \) and computes a preventive maintenance cost \( TC_{MAI} \) associated to the inspection design vector \( \Omega = \{ I_s, 0, 0, 0 \} \):

\[
TC_{MAI} = E(TC|I_s, n = 0) = M \cdot \left( h - \frac{\gamma}{\nu} \right)(I_s + 1) + \left[ L_0 \cdot (1 - \gamma) + L_4 \cdot \gamma \right] I_s. \tag{14}
\]

If the minimum preventive maintenance cost is less than \( TC_{NC} \), then the preventive maintenance is selected as the current inspection strategy. The optimization continues by varying the design variables of the CUSUM \( t \) chart within their allowable ranges and computing \( E(TC) \) through equation (9). The algorithm stops when all the feasible designs have been investigated and selects the optimal design \( \Omega^* \) that leads to the minimum quality control cost resulting from the no control, the preventive maintenance or the control chart implementation. A step-by-step description of the exhaustive optimization algorithm is presented in Figure 2.

4. Numerical Results

The benchmark of 64 scenarios investigated in Nenes and Tagaras [12] has been run here to test the economic performance of the CUSUM \( t \) chart. The following six parameters are varied at two levels: the production horizon length \( H \in \{8; 40\} \) hours; the shift size \( \delta \in \{0.5; 1\} \); the failure rate \( \nu \in \{0.01; 0.05\} \) failures/hour; the fixed sampling cost \( a \in \{0; 5\} \) $; the out-of-control loss rate \( M \in \{100; 1000\} \) $/hour; the search/false alarm cost \( L_0 \in \{50; 150\} \) $.
Finally, the variable sampling cost \(b\) has been fixed equal to 1$/part and the “search + restoration” cost \(L_1\) has been set equal to 150$. Without loss of generality, an unitary group size \(B = 1\) has been considered to evaluate the CUSUM \(t\) chart’s performance within the largest set of feasible design solutions. Table 2 shows the investigated benchmark of scenarios.

\[
\text{Variables initialization} \\
\text{Set } \Omega = \{0, 0, 0, 0\}, \Omega' = \Omega \text{ and } TC_{\text{best}} = 0
\]

\[
\text{No control strategy} \\
\text{Compute } TC_{\text{NC}} \text{ and Set } TC_{\text{best}} = TC_{\text{NC}} \quad \% TC_{\text{NC}} \text{ is computed through eqn. (13)}
\]

\[
\text{Preventive maintenance strategy optimization} \\
\text{[cycle } l_s \text{] Do } l_s = 1, 2,\ldots, N/B \\\n\text{Set } \Omega = \{l_s, 0, 0, 0\} \text{ and Compute } TC_{\text{MAI}}(l_s) \quad \% TC_{\text{MAI}} \text{ is computed through eqn. (14)}
\]

\[
\text{If } TC_{\text{MAI}}(l_s) < TC_{\text{best}} \text{ Then} \\
TC_{\text{best}} = TC_{\text{MAI}}(l_s) \\
\Omega' = \Omega
\]

\[
l_s = l_s + 1
\]

\[
\text{End Do [cycle } l_s \text{]}
\]

\[
\text{CUSUM chart optimization} \\
\text{[cycle } l_s \text{] Do } l_s = 1, 2,\ldots, N/B \\
\text{[cycle } n \text{] Do } n = 1, 2,\ldots, n_{\text{max}} \quad \% n_{\text{max}} \text{ is constrained by eqn. (1)}
\]

\[
\text{[cycle } UCL \text{] Do } UCL = 0.05, 0.1,\ldots, 5 \\
\text{[cycle } k \text{] Do } k = 0.1, 0.2,\ldots, 2 \\
\text{Set } \Omega = \{l_s, n, UCL, k\} \\
\text{Compute } E(TC(\Omega)) \quad \% E(TC) \text{ is computed through eqn. (9)}
\]

\[
\text{If } E(TC(\Omega)) < TC_{\text{best}} \text{ Then} \\
TC_{\text{best}} = E(TC(\Omega)) \\
\Omega' = \Omega
\]

\[
k = k + 0.1
\]

\[
\text{End Do [cycle } k \text{]}
\]

\[
UCL = UCL + 0.05
\]

\[
\text{End Do [cycle } UCL \text{]}
\]

\[
n = n + 1
\]

\[
\text{End Do [cycle } UCL \text{]}
\]

\[
l_s = l_s + 1
\]

\[
\text{End Do [cycle } l_s \text{]}
\]

\[
\text{Stop}
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Figure 2. Step-by-step description of the exhaustive optimization algorithm.
Table 2. The benchmark of investigated scenarios (reprinted from Nenes and Tagaras [12]).

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For each scenario out of the benchmark, the economic design has been selected under the following two conditions: (i) optimization constrained by the production and inspection rates, i.e. \( r_{PR} = 10 \) parts/hour and \( r_{I} = 10 \) parts/hour: this condition can occur in those manufacturing environments where several operations are requested on a part at a workstation and the downstream inspection procedure takes a long time to get the sample measures; (ii) optimization unconstrained by the production and inspection rates, i.e. \( r_{PR} = 100 \) parts/hour and \( r_{I} = 100 \) parts/hour: high values of \( r_{PR} \) and \( r_{I} \) are frequent in the intensive production and inspection of simple parts. Tables 3 and 4 show the economic design and the expected quality control cost \( E(TC) \) associated with the CUSUM \( t \) chart and CUSUM \( X \) chart with known parameters for both the constrained and unconstrained optimizations.

In the same tables, the number of inspections \( I_s = 0 \) for a process scenario means that “no monitoring” is the most convenient inspection strategy. “No monitoring” is found to be the optimal inspection solution for 9 scenarios, for both the CUSUM \( t \) and CUSUM \( X \) chart. All of these manufacturing scenarios are characterized by a very short rolling horizon \( H = 8 \) hours, a low failure rate \( \nu = 0.01 \) (with the exception of one scenario) and low out-of-control loss rate \( M = 100 \) $/hour. This combination of process cost and operating parameters allows for the inspection relaxation due to the negligible cost impact of the out-of-control production condition vs. the effect that costs related to sampling and inspecting parts have on \( E(TC) \). It is worth noting that the same results are obtained under both the conditions investigated in Tables 3 and 4: in fact, they are not influenced either by the production rate or the inspection rate constraint.

Table 3 also shows that the constraints about process productivity and the inspection equipment performance limit to \( n_{max} \) the selection of the optimal sample size for 18 scenarios, (these cases are highlighted by showing bold and underlined values for \( n \)). This conclusion is
applicable to both the charts, but in 9 out of 18 scenarios the CUSUM \( t \) chart is designed with larger sample sizes and less scheduled inspections than the CUSUM \( \bar{X} \) chart. As expected, this result demonstrates that the chart based on the \( t \) statistic needs to have larger sample sizes than the one for monitoring the \( \bar{X} \) statistic due to its lower sensitivity to the process shift. This finding resembles what happens to the hypothesis testing procedures with unknown parameters, which usually have a larger Type II error than the test with known parameters for the same sample size. This trend is also confirmed in Tables 3 and 4 for those scenarios not constrained by \( n_{\text{max}} \).

If we denote as the sampling intensity \( SI = 100 \cdot n_{\text{I}}/N = 100 \cdot n_{\text{I}}/(H \cdot r_{\text{PR}}) \) the percentage of produced items to be inspected during the production run, \( SI \) is roughly the same for the two charts: in fact, for the constrained optimization, the averaged value of \( SI \) over the benchmark is equal to 43.4\% for the CUSUM \( t \) chart and 42.8\% for the CUSUM \( \bar{X} \) chart, whereas the same maximum value of 90\% is encountered for both the charts. Similarly, for the unconstrained optimization the averaged value of \( SI \) over the benchmark is equal to 5.9\% for the CUSUM \( t \) chart and 5.8\% for the CUSUM \( \bar{X} \) chart; the same maximum value of \( SI \) equal to 22\% is encountered for both the charts. For the two optimizations, the large difference between the averaged values of \( SI \) is due to the production and inspection rates: in fact, in the unconstrained optimization such a large values of \( n_{\text{max}} \) and/or \( I_{\text{I}} \) are allowed that the selected economic design parameters represent only a small fraction of them.

Table 3. Optimal designs of the CUSUM \( t \) chart in comparison with the CUSUM \( \bar{X} \) chart, \((r_{\text{PR}} = 10 \text{ parts/hour} \text{ and } n = 10 \text{ parts/hour})\).
Constrained condition, where the sampling intensity can have a large influence on the investigated; this step of the numerical study is important especially with reference to the constrained condition, where the sampling intensity can have a large influence on the.

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<td>0.85</td>
<td>779.30</td>
<td>4.72</td>
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<td>56</td>
<td>6</td>
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<td>0.65</td>
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<tr>
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<td>9</td>
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<td>0.65</td>
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<td>7.67</td>
</tr>
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</tr>
<tr>
<td>63</td>
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<td>0.55</td>
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</tr>
<tr>
<td>64</td>
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<td>1.3</td>
<td>0.55</td>
<td>2371.27</td>
<td>5.57</td>
</tr>
</tbody>
</table>

The variation of SI throughout the benchmark of scenarios is also worth to be investigated: this step of the numerical study is important especially with reference to the constrained condition, where the sampling intensity can have a large influence on the.
requested SPC resource capacity. In what follows, reference will be implicitly done to the implementation of the CUSUM $t$ chart, because the findings for this chart are very close to the CUSUM $\bar{X}$ chart. A statistical analysis investigating the effect of the varying factors on the response variable $SI$ has been performed on the benchmark: the ANOVA table and the fitted response surface generated by the Design Expert® software are presented in Figure 2. Due to the absence of experimental error for the investigated problem, the noise term of the ANOVA has been computed through the not influencing higher order terms: this approach has been adopted in similar studies by Simpson and Keats [15], Castagliola et al. [2], Celano [3]. The model checking of the ANOVA residuals did not show any anomaly.

Table 4. Optimal designs of the CUSUM $t$ chart in comparison with the CUSUM $\bar{X}$ chart, ($r_{PR} = 100$ parts/hour and $n = 100$ parts/hour).

| Prob. | $n_{max}$ | $I$ | $n$ | $k$ | UCL | $E(TC)$ | $n_{max}$ | $I$ | $n$ | $k$ | UCL | $E(TC)$ | C% |
|-------|-----------|-----|-----|-----|-----|--------|-----------|-----|-----|-----|-----|-----|-----|-----|
| 1     | 799       | 0   | -   | -   | 31.16 | 799     | 0   | -   | -   | 31.16 | 0.00 |
| 2     | 799       | 0   | -   | -   | 31.16 | 799     | 0   | -   | -   | 31.16 | 0.00 |
| 3     | 199       | 3   | 11  | 0.3 | 0.55 | 178.76  | 199     | 3   | 11  | 0.4 | 0.45 | 177.64 | 0.63 |
| 4     | 265       | 2   | 25  | 1.0 | 0.55 | 208.07  | 265     | 2   | 23  | 0.9 | 0.55 | 205.87 | 1.07 |
| 5     | 799       | 0   | -   | -   | 31.16 | 799     | 0   | -   | -   | 31.16 | 0.00 |
| 6     | 799       | 0   | -   | -   | 31.16 | 799     | 0   | -   | -   | 31.16 | 0.00 |
| 7     | 265       | 2   | 13  | 0.3 | 0.35 | 192.86  | 265     | 2   | 14  | 0.4 | 0.35 | 192.31 | 0.29 |
| 8     | 265       | 2   | 25  | 1.0 | 0.55 | 218.07  | 265     | 2   | 23  | 0.9 | 0.55 | 215.87 | 1.02 |
| 9     | 399       | 1   | 9   | 0.1 | 0.65 | 129.44  | 399     | 1   | 9   | 0.7 | 0.05 | 129.12 | 0.25 |
| 10    | 399       | 1   | 14  | 1.0 | 0.55 | 139.22  | 399     | 1   | 14  | 0.9 | 0.65 | 137.97 | 0.91 |
| 11    | 79        | 9   | 11  | 0.3 | 0.65 | 465.13  | 71      | 10  | 10  | 0.3 | 0.75 | 461.24 | 0.84 |
| 12    | 99        | 7   | 21  | 0.8 | 0.75 | 550.50  | 99      | 7   | 21  | 0.9 | 0.65 | 542.55 | 1.47 |
| 13    | 399       | 1   | 9   | 0.1 | 0.65 | 134.44  | 399     | 1   | 9   | 0.7 | 0.05 | 134.12 | 0.24 |
| 14    | 799       | 0   | -   | -   | 140.64 | 799     | 0   | -   | -   | 140.64 | 0.00 |
| 15    | 99        | 7   | 12  | 0.3 | 0.45 | 503.95  | 99      | 7   | 12  | 0.3 | 0.55 | 501.93 | 0.40 |
| 16    | 132       | 5   | 26  | 0.9 | 0.55 | 580.65  | 113     | 6   | 23  | 0.9 | 0.45 | 574.34 | 1.10 |
| 17    | 799       | 0   | -   | -   | 31.16 | 799     | 0   | -   | -   | 31.16 | 0.00 |
| 18    | 799       | 0   | -   | -   | 31.16 | 799     | 0   | -   | -   | 31.16 | 0.00 |
| 19    | 132       | 5   | 8   | 1.1 | 0.65 | 132.49  | 113     | 6   | 6   | 0.9 | 0.85 | 125.31 | 5.73 |
| 20    | 159       | 4   | 12  | 1.6 | 0.65 | 148.14  | 132     | 5   | 9   | 1.4 | 0.75 | 138.92 | 6.64 |
| 21    | 799       | 0   | -   | -   | 31.16 | 799     | 0   | -   | -   | 31.16 | 0.00 |
| 22    | 799       | 0   | -   | -   | 31.16 | 799     | 0   | -   | -   | 31.16 | 0.00 |
| 23    | 199       | 3   | 9   | 1.1 | 0.45 | 151.85  | 159     | 4   | 8   | 1.1 | 0.55 | 147.58 | 2.89 |
| 24    | 199       | 3   | 13  | 1.6 | 0.55 | 164.15  | 199     | 3   | 11  | 1.5 | 0.55 | 157.70 | 4.09 |
| 25    | 265       | 2   | 7   | 1.3 | 0.25 | 116.13  | 265     | 2   | 7   | 1.3 | 0.25 | 113.58 | 2.25 |
| 26    | 399       | 1   | 12  | 1.9 | 0.15 | 121.58  | 265     | 2   | 9   | 1.6 | 0.45 | 119.08 | 2.10 |
| 27    | 60        | 12  | 8   | 1.1 | 0.65 | 344.16  | 49      | 15  | 6   | 1.0 | 0.75 | 325.91 | 5.60 |
| 28    | 71        | 10  | 12  | 1.5 | 0.75 | 384.60  | 60      | 12  | 9   | 1.3 | 0.85 | 360.55 | 6.67 |
| 29    | 399       | 1   | 9   | 1.0 | 0.45 | 122.48  | 399     | 1   | 8   | 0.9 | 0.45 | 121.42 | 0.87 |
| 30    | 399       | 1   | 12  | 1.9 | 0.15 | 126.58  | 399     | 1   | 11  | 0.9 | 1.05 | 124.56 | 1.62 |
| 31    | 79        | 9   | 9   | 1.1 | 0.55 | 394.57  | 79      | 9   | 8   | 1.1 | 0.45 | 383.21 | 2.96 |
| 32    | 87        | 8   | 13  | 1.6 | 0.55 | 427.90  | 79      | 9   | 11  | 1.4 | 0.75 | 410.67 | 4.20 |
| 33    | 570       | 6   | 10  | 0.2 | 0.75 | 380.79  | 570     | 6   | 10  | 0.3 | 0.65 | 325.79 | 0.74 |
| 34    | 799       | 4   | 23  | 0.9 | 0.65 | 380.79  | 799     | 4   | 21  | 0.9 | 0.55 | 376.56 | 1.12 |
Figure 3 (upper part) shows the ANOVA table for the $SI$ response variable: about 93% of $SI$ variability is explained by the out-of-control loss rate $M$, the failure rate $v$ and the length of the production run $H$. Figure 3 (lower part) gives a visual appreciation about how $SI$ is expected to vary vs. $(M, H)$ within the investigated region: the largest values of $SI$ occur when high loss rates and longer rolling horizons are considered; for example, at the high level of the loss rate ($M = 1000$ $$/hour) SI$ varies between 70% and 90%; otherwise, when the loss rate ($M = 100$ $$/hour) is at the low level $SI$ approximately varies between 20% and 35%. Clearly, a point on the surface is an expected $SI$ prediction, but the surface is a good aid to visualize the trends in the $SI$ variation. It is worth noting that the highest values of the sampling intensity $SI$ occur for those scenarios where the inspection design is constrained by the maximum sample size $n_{max}$: from a practical point of view, this suggests to quality practitioners to be aware about the process configuration constraints when designing inspection procedures for very short run processes on highly technological parts characterized by slow production and inspection rates and high loss rates when producing in the out-of-control state.
For the sake of brevity, the same statistical analysis is not reported here for the *unconstrained condition*, whose trends resemble the previous case apart from their absolute value. About 81% of $SI$ variability is mainly explained by the out-of-control loss rate $M$, the failure rate $\nu$ and their interaction $M \cdot \nu$; furthermore, the false alarm cost $L_0$ and the length of the production run $H$ have a slightly significant influence (about 3% of $SI$ total variability within the benchmark is influenced by each of them).

Another important issue worth to be investigated is the economic loss associated with the $t$ monitoring statistic when population parameters cannot be estimated vs. the “ideal”
condition of perfectly known parameters. To do this, the last columns of Tables 3 and 4 show the percentage loss $\Delta C\%$ associated with the implementation of the CUSUM $t$ chart vs. the CUSUM $X$ chart with known parameters, which is defined as:

$$\Delta C\% = 100 \frac{E(TC) - E(TC)_X}{E(TC)_X}$$

(15)

Large values of $\Delta C\%$ mean a significant cost loss associated to the $t$ chart. Of course, a loss due to the lack of knowledge about the population distribution was expected, but, quite surprisingly, the average $\Delta C\%_{ave}$ over the benchmark is equal to 2.25% (2.23%) for the constrained (unconstrained) condition. Similarly, the maximum value is less than 8% for both the conditions. Furthermore, 49 out of 64 scenarios have a $\Delta C\%$ of less than 4%. From a practical point of view, this finding looks really promising because it proves the CUSUM $t$ chart as an economically efficient solution to the problem of performing SPC in a short run where no knowledge about the quality characteristic can be gathered either in advance or through a Phase I implementation of the chart. Given these results, the variability of $\Delta C\%$ is also worth to be investigated. Thus, a statistical analysis has been carried out by assuming $\Delta C\%$ as the response variable of the 26 experimental design derived from the benchmark of scenarios. Once again, for the sake of brevity only the output for the constrained condition generated by Design Expert is presented in Figure 4.

The residual analysis did not show any particular anomaly. The ANOVA table shows that $\delta$, $M$, $a$ and $H$ are in order of magnitude the influencing factors on $\Delta C\%$. All the main factors, except the fixed sampling cost $a$, have a positive effect on $\Delta C\%$. The shift size $\delta$ accounts for about 50% of the $\Delta C\%$ variation in the benchmark of cases: those subset of 32 scenarios optimized for an expected smaller shift size ($\delta=0.5$) have an average value of $\Delta C\%_{ave}$ = 0.65%, whereas it results in $\Delta C\%_{ave}$ = 3.86% for the remaining subset of 32 scenarios having ($\delta = 1.0$). Thus, the economic loss gets a little bit larger once the economic design is selected for larger process shifts: this probably depends on a steeper rate of sensitivity increase vs. the shift size performed by the control chart when the parameters are known. The out-of-control loss rate $M$ is the second most important influencing factor, (it accounts for 14.3% of the $\Delta C\%$ variation in the benchmark).

Its positive effect on $\Delta C\%$ reflects the influence of the chart’s sensitivity on the economic loss: a longer expected out-of-control production length than the CUSUM $X$ chart with known parameters is expected when the CUSUM $t$ chart is implemented due to its lower sensitivity to process shifts; thus, higher out-of-control loss rates increase the cost difference between the two charts. The fixed sampling cost $a$ has a marginal negative effect on the $\Delta C\%$, because it adds to the expected quality control cost during the run a fixed cost proportional to the number $I_t$ of scheduled inspections. Finally, the time length $H$ of the production run has a positive effect on $\Delta C\%$: depending on the chart’s sensitivity, a signal about the occurrence of an out-of-control condition could not be immediately triggered, thus longer production runs could be characterized by longer out-of-control productions and higher economic losses. The statistically significant second order interactions $M\delta$ and $\delta H$ confirm the discussion above attributing the largest part of the $\Delta C\%$ economic loss to the difference in statistical sensitivity between the two charts. Table 5 summarizes the influencing factors and the signs of their effects for both the $SI$ and $\Delta C\%$ response variables.
Figure 4. Investigation of the ΔC% variation: ANOVA Table (upper part) and fitted Response Surface (lower part) for ν=0.05 failures/h, a=5$, L_0=150$ and H=40 hours.

Table 5. Influencing process parameters and sign (+/-) of their effects for the Sampling Intensity SI and the Economic Loss C%

<table>
<thead>
<tr>
<th></th>
<th>SI</th>
<th>C%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unconstrained</td>
<td>Constrained</td>
<td>Unconstrained</td>
</tr>
<tr>
<td>M (+)</td>
<td>M (+)</td>
<td>δ (+)</td>
</tr>
<tr>
<td>ν (+)</td>
<td>ν (+)</td>
<td>M (+)</td>
</tr>
<tr>
<td>M · ν (+)</td>
<td>H (+)</td>
<td>M · δ (+)</td>
</tr>
<tr>
<td>L_0 (+)</td>
<td>H (+)</td>
<td>δ · H (+)</td>
</tr>
<tr>
<td>H (+)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Often, quality practitioners do not implement the economic optimization of the design parameters of a control chart or an alternative inspection strategy because it is difficult to get a precise estimate of each process cost and operating parameter to be included in the model. Thus, an investigation about the effect of the misspecification of these parameters is worth to be carried out. Following the same approach as in Nenes and Tagaras [12], a one-at-a-time sensitivity analysis on the misspecification of $\delta$, $v$, $M$, $L_0$, $L_1$, $a$ and $b$ is performed for the constrained optimization. An overestimation/underestimation error equal to $\pm 20\%$ in one parameter selection has been assumed for each scenario out of the benchmark and the economic design of the inspection strategy has been obtained through the optimization procedure. For each scenario, the sub-optimal quality control cost corresponding to this economic design and the correct parameter value is evaluated and compared with the minimum $E(TC)$ presented in Table 3. A penalty cost can be defined as the percentage increase of the sub-optimal quality control cost vs. the minimum quality control cost. The average and maximum penalty costs for the benchmark and corresponding to each parameter misspecification are presented in Table 6.

The obtained results are similar to those reported in Nenes and Tagaras [12] and demonstrate that implementing a CUSUM $t$ chart does not affect the sensitivity of the economic design selection to the underestimation / overestimation of the process cost and operating parameters vs. the CUSUM $\bar{X}$ chart. In fact, the quality control function flatness in the region of the minimum cost provides quality practitioners with a sufficient robustness vs. the estimation of the parameters: the larger average penalty cost occurs with an underestimation of the shift size $\delta$ and is lower than $1.7\%$. All the remaining parameters do not have an important effect on the design selection. The maximum penalty cost occurs for an overestimation of $M$ and is equal to $6.64\%$. Finally, Table 6 also presents the effect of the length $H$ of the production horizon: a paired $t$-test with $\alpha = 0.05$ has been conducted on the two populations each consisting of 32 scenarios having $H = 8, 40$. The hypothesis test resulted significant for $\delta$, $L_0$ and $b$.

<table>
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<tr>
<th>Average Penalty Cost (%)</th>
<th>Maximum Penalty Cost (%)</th>
<th>Production Horizon Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U / O$</td>
<td>$U / O$</td>
<td>$U / O$</td>
</tr>
<tr>
<td>$\delta$</td>
<td>1.62 / 1.27</td>
<td>4.48 / 3.30</td>
</tr>
<tr>
<td>$v$</td>
<td>0.48 / 0.26</td>
<td>3.18 / 2.60</td>
</tr>
<tr>
<td>$M$</td>
<td>0.65 / 0.54</td>
<td>4.61 / 6.64</td>
</tr>
<tr>
<td>$L_0$</td>
<td>0.30 / 0.28</td>
<td>1.07 / 2.22</td>
</tr>
<tr>
<td>$L_1$</td>
<td>0.10 / 0.10</td>
<td>2.60 / 1.20</td>
</tr>
<tr>
<td>$a$</td>
<td>0.05 / 0.06</td>
<td>0.90 / 0.90</td>
</tr>
<tr>
<td>$b$</td>
<td>0.38 / 0.22</td>
<td>2.20 / 1.02</td>
</tr>
</tbody>
</table>

5. An Illustrative Example

An illustrative example is now detailed to give readers more insight about the proposed constrained economic model. A manufacturing company produces by means of a CNC
machining center customized high-tech mechanical parts having a complex shape: a quality characteristic of these components is their surface roughness on one specific surface. The nominal average roughness (target) is \( R_a = U = 3.00 \, \mu m \), the specification width is equal to 60.2 \( \mu m \). The variation of the average profile roughness among different parts can be influenced by the occurrence of vibrations due to an imperfect fixture of parts on the CNC center work table. On the CNC machine, parts are loaded one by one, thus \( B = 1 \) part. A short production run has been scheduled to manufacture \( N=100 \) parts. As the time required to load/unload and work a part on the CNC center is 15 min, the rate of production is equal to \( r_{PR} = 4 \) parts/hour. Under the simplifying assumption of full process efficiency, the production length is assumed to be equal to \( H = N/r_{PR} = 25 \) hours.

The downstream SPC inspection activities on the surface roughness are performed locally at the CNC center and require 5 min to be measured by means of a manual profilometer: thus, it holds \( r_i = 12 \) parts/hour. The hourly cost of the worker is \( c_{LR} = 36 \) $/h. The fixed sampling cost \( a \) is equal to 15$; as the sample measure is not destructive, the variable sampling cost is computed as the unit cost to get a measure, i.e. \( b = c_{LR}/r_i = 3 \) $/part. The cost to search for a false alarm is \( L_0 = 25 \) $ and accounts for the repairing crew assembly and operation and the necessary search equipment. The cost to search and restore the process to the in-control condition is \( L_1 = 50 \) $. The out-of-control loss rate \( M \) is equal to 400$ /hour and accounts for the costs associated with reworking non-conforming parts at the grinding machine or their rejection and the costs related to possible returns/complaints from the customer. Under the assumption of an initial perfect tooling set-up and a correct fixture of the part on the CNC center work table, the in-control population mean \( \mu_0 \) can be considered approximately centred on the target \( U \), i.e. \( \mu_0 = 3 \mu m \). The in-control standard deviation \( \sigma_0 \) is unknown.

A CUSUM \( \tau \) control chart is a good candidate to monitor such a manufacturing process. To show its implementation, its design parameters have been selected by considering a process failure rate \( \nu = 0.05 \) failures/hour and an expected shift size \( \delta = 1 \). The economic design of the CUSUM \( \tau \) chart for the investigated process schedules \( I_0 = 18 \) inspections of \( n = 4 \) parts within the production run, (for \( I_0 = 18 \) and the input values of \( H, B, r_{PR} \) and \( r_i \), the maximum feasible sample size was \( n_{\text{max}} = 4 \), accordingly to equation (1)). Therefore, the sampling intensity is \( SI = 72\% \). The other selected design parameters are the reference value \( k = 0.3 \) and the upper control limit \( UCL = 0.45 \). A quality control cost \( E(TC) \) equal to 677.21$ is expected during the run. To show the implementation of the CUSUM \( \tau \) chart, the SPC inspection of the parts during the production run has been simulated as follows: the process operates in the in-control state \( (Y_i = 0) \) for 12 hours; then, it shifts to the out-of-control state \( (Y_i = 1) \): a special cause occurs between the 9th and 10th scheduled inspection and shifts up the process mean with a shift size equal to \( \delta = 1 \).

Table 7 shows the measures collected to perform the SPC inspections during the production run. The CUSUM \( \tau \) chart is presented in Figure 5.

Inspections \#1-\#9 have been simulated by randomly generating the measures as normally distributed random numbers \( N(3.00, 0.05) \). Then, the inspections \#10-\#11 collected before the chart signals have been simulated by randomly generating the measures as normally distributed random numbers \( N(3.05, 0.05) \). In Table 7, the numbers in italic refer to the out-of-control condition.
Table 7. An illustrative example: simulated measures and collected statistics for the CUSUM $t$ chart.

<table>
<thead>
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<th>Insp. number</th>
<th>Insp. Time [h]</th>
<th>$X_{i1}$</th>
<th>$X_{i2}$</th>
<th>$X_{i3}$</th>
<th>$X_{i4}$</th>
<th>$t_i$</th>
<th>$C_i$</th>
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<td>17</td>
<td>3.045</td>
<td>2.928</td>
<td>2.992</td>
<td>2.989</td>
<td>-0.245</td>
<td>0.000</td>
</tr>
<tr>
<td>0.000</td>
<td>18</td>
<td>2.976</td>
<td>3.003</td>
<td>2.981</td>
<td>2.979</td>
<td>-1.220</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Figure 5. The implementation of the CUSUM $t$ chart.

The CUSUM $t$ chart triggers a signal at inspection #11, i.e. at the second sample after the process shifts. Thus, the “search+correction” activities are started as soon as possible. For simplicity, in this example we have assumed that restoring the process to the out-of-control condition does not delay either the future inspection activities or the end of the production run: of course, delays are likely to occur when interventions are required on a machine, thus
stopping the manufacturing operations. After the process is restored to the in-control condition and without the occurrence of any further process shift, inspections #12-#18 have been simulated by randomly generating the measures as normally distributed random numbers $N(3.00, 0.05)$. The process continues up to the end of production run without any more chart’s signal.

6. Conclusions

When dealing with short production runs, designing on-line SPC inspection activities can be difficult because of the lack of previous knowledge about the distributional properties of the quality characteristic to be monitored. In fact, a correct estimation of the distribution parameters through a *Phase I* implementation of a control chart could not be possible due to the run’s shortness and/or slow production rates and/or inspection rates. In this paper, to monitor the process mean in a short run, the CUSUM $t$ control chart and its economic design have been proposed to overcome the problem of the preliminary estimation of the distribution parameters. The proposed chart allows for an immediate start of the SPC inspection activities at the beginning of the production run and at convenient cost.

In terms of economic loss with respect to the limiting condition of a chart’s implementation with perfectly known parameters, the results presented in this paper look promising. In fact, the economic loss always remains at an acceptable level and becomes negligible for those strategic manufacturing environments where small batches of customized high-tech products should be produced in flexible manufacturing cells or machines designed to carry out complex operations. In these manufacturing environments slow production and inspection rates can occur, which limit the space of feasible chart design solutions: thus, quality practitioners are advised to take care about the process operating conditions by introducing them as constraints of the optimization.

Of course, further research is needed to improve the efficiency of the proposed chart. In particular, as demonstrated by previous literature, see Celano *et al.* [4], the $t$ charts sensitivity in detecting a shift in the process mean suffers from the unexpected occurrence of a shift in the process dispersion. Thus, priority should be done to research topics focusing on designing tools able to start the process dispersion monitoring at the beginning of the short run without any previous knowledge about the quality characteristic. Furthermore, for the sake of brevity only the perfect initial setup condition has been investigated in this paper: a systematic study about the effect of an imperfect setup on the quality control cost and chart’s performance would be interesting. Finally, other control chart schemes like the time weighted EWMA charts or adaptive charts are equally worth to be investigated; the extension to the multivariate chart is also an interesting topic to be challenged by researchers.

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References


Appendix

By combining the actual process state $Y_i = y$ with $y \in \{0, 1\}$ and the value of the statistic $C_i$ for $i = 0, \ldots, I$, and taking into account the exponential failure mechanism of the process, the couple $(Y_i, C_i)$ can be represented by a two-dimensional Markov chain, see Nenes and Tagaras [12]. In order to follow the Brook and Evans [1] approach, the continuous random variable $C_i$ is discretized into $m + 1$ values, i.e. the interval between 0 to $UCL$ is divided into $m$ sub-intervals. The width $\rho$ of a generic sub-interval is derived as follows:

$$\rho = \frac{2 \cdot UCL}{2m - 1}, \quad m = \frac{UCL}{\rho} + \frac{1}{2}. \quad \text{(A.1)}$$

As a consequence, $C_i$ is quantized as it takes the following discrete values:

$$C_i = \begin{cases} 
C_0 = 0 & \text{for } 0 \leq C_i \leq \frac{\rho}{2} \\
j\rho & \text{for } \left( j - \frac{1}{2} \right) \rho \leq C_i \leq \left( j + \frac{1}{2} \right) \rho \\
UCL & \text{for } C_i > \left( m - \frac{1}{2} \right) \rho 
\end{cases} \quad \text{(A.2)}$$

where $j\rho$ represents the midpoint of the generic sub-interval and $j \in [1, \ldots, m]$. Consequently, as $Y_i$ can assume the values $[0, 1]$, the Markov chain $(Y_i, C_i)$ has $2 \times (m + 1) = 2m + 2$ states and the $(2m + 2) \times (2m + 2)$ transition probability matrix $P$ takes the following form:

$$P = \begin{bmatrix}
Y_i = 0 & Y_i = 1 \\
v = 0 & v = 1 & \ldots & v = m \\
u = 0 & u = 1 & \ldots & u = m \\
Y_{i-1} = 0 & \ldots & Y_{i-1} = 1 \\
u = m & - & - & - & \ldots & - & - & - & - \\
u = 0 & - & - & - & - & - & - & - & - \\
Y_{i-1} = 1 & u = 1 & - & - & - & - & - & - & - & - \\
u = m & - & - & - & - & - & - & - & - & - 
\end{bmatrix} \quad \text{(A.3)}$$

The elements of $P$ are grouped into four sub-matrices representing the one-step transition probabilities of moving from $C_{i-1}$ to $C_i$ for each of the possible combinations of $Y_{i-1}$ and $Y_i$:

$$p_{u,v} = P\left[ C_i = v, Y_i = 0 \mid C_{i-1} = u, Y_{i-1} = 0 \right]$$

$$q_{u,v} = P\left[ C_i = v, Y_i = 1 \mid C_{i-1} = u, Y_{i-1} = 0 \right]$$

$$r_{u,v} = P\left[ C_i = v, Y_i = 0 \mid C_{i-1} = u, Y_{i-1} = 1 \right]$$

$$s_{u,v} = P\left[ C_i = v, Y_i = 1 \mid C_{i-1} = u, Y_{i-1} = 1 \right], \quad u, \ v = 1, \ldots, m. \quad \text{(A.4)}$$
Considering the random shock exponential model assumed for the process, the transition probabilities can be computed as follows:

\[
\begin{align*}
  p_{u,v} &= (1 - \gamma) \cdot F_i \left( \left( \frac{1}{2} - u \right) \rho + k \left| n - 1 \right| \right) & u = 0, \ldots, m-1 \quad v = 0 \\
  &+ (1 - \gamma) \cdot \left[ F_i \left( \left( v + \frac{1}{2} - u \right) \rho + k \left| n - 1 \right| \right) - F_i \left( \left( v - \frac{1}{2} - u \right) \rho + k \left| n - 1 \right| \right) \right] & u = 0, \ldots, m-1 \\
  &+ (1 - \gamma) \cdot \left[ 1 - F_i \left( UCL - u \rho + k \left| n - 1 \right| \right) \right] & u = 0, \ldots, m-1 \\
  p_{m,v} &= p_{0,v} & u = m \quad v = 0, 1, \ldots, m
\end{align*}
\]

(A.5)

\[
\begin{align*}
  q_{u,v} &= \gamma \cdot G_i \left( \left( \frac{1}{2} - u \right) \rho + k - \delta \sqrt{n} \left| n - 1, \delta \sqrt{n} \right| \right) & u = 0, \ldots, m-1 \quad v = 0 \\
  &+ \gamma \cdot \left[ G_i \left( \left( v + \frac{1}{2} - u \right) \rho + k - \delta \sqrt{n} \left| n - 1, \delta \sqrt{n} \right| \right) - G_i \left( \left( v - \frac{1}{2} - u \right) \rho + k - \delta \sqrt{n} \left| n - 1, \delta \sqrt{n} \right| \right) \right] & u = 0, \ldots, m-1 \\
  &+ \gamma \cdot \left[ 1 - G_i \left( UCL - u \rho + k - \delta \sqrt{n} \left| n - 1, \delta \sqrt{n} \right| \right) \right] & u = 0, \ldots, m-1 \\
  q_{m,v} &= q_{0,v} & u = m \quad v = 0, 1, \ldots, m
\end{align*}
\]

(A.6)

\[
\begin{align*}
  r_{u,v} &= (1 - \gamma) \cdot F_i \left( \frac{\rho}{2} + k \left| n - 1 \right| \right) & u = m \quad v = 0, \ldots, m \\
  &+ (1 - \gamma) \cdot \left[ F_i \left( \left( v + \frac{1}{2} \right) \rho + k \left| n - 1 \right| \right) - F_i \left( \left( v - \frac{1}{2} \right) \rho + k \left| n - 1 \right| \right) \right] & u = m \quad v = 0, \ldots, m-1 \\
  &+ (1 - \gamma) \cdot \left[ 1 - F_i \left( UCL + k \left| n - 1 \right| \right) \right] & u = m \quad v = m
\end{align*}
\]

(A.7)
\[ s_{u,v} = \begin{cases} 
G_i \left( \frac{1}{2} - u \right) \rho + k - \delta \sqrt{n} \big| n-1, \delta \sqrt{n} \big) & u = 0, \ldots, m-1 \quad v = 0 \\
\left[ G_i \left( \nu + \frac{1}{2} - u \right) \rho + k - \delta \sqrt{n} \big| n-1, \delta \sqrt{n} \big) - G_i \left( \nu - \frac{1}{2} - u \right) \rho + k - \delta \sqrt{n} \big| n-1, \delta \sqrt{n} \big) \right] & u = 0, \ldots, m-1 \quad v = 0, \ldots, m-1 \\
1 - G_i \left( \text{UCL} - u \rho + k - \delta \sqrt{n} \big| n-1, \delta \sqrt{n} \big) \right) & u = 0, \ldots, m-1 \quad v = m \\
\gamma \cdot G_i \left( \frac{\rho}{2} + k - \delta \sqrt{n} \big| n-1, \delta \sqrt{n} \big) \right) & u = m \quad v = 0 \\
\gamma \cdot G_i \left( \nu + \frac{1}{2} \right) \rho + k - \delta \sqrt{n} \big| n-1, \delta \sqrt{n} \big) \right) - G_i \left( \nu - \frac{1}{2} \right) \rho + k - \delta \sqrt{n} \big| n-1, \delta \sqrt{n} \big) \right) & u = m \quad v = 0, \ldots, m-1 \\
\gamma \cdot 1 - G_i \left( \text{UCL} + k - \delta \sqrt{n} \big| n-1, \delta \sqrt{n} \big) \right) & u = m \quad v = m 
\end{cases} \]

(A.8)

where \( F_i \left( \nu \big| n-1 \right) \) denotes the Student \( t \) cumulative distribution function with \( n-1 \) degrees of freedom and \( G_i \left( \nu \big| n-1, \psi = \delta \sqrt{n} \right) \) represents the non-central Student \( t \) cumulative distribution function with \( n-1 \) degrees of freedom and non-centrality parameter \( \delta \sqrt{n} \). It is worth noting that when the chart signals an out-of-control condition, the stochastic model assumes a procedure calling for the search of the special cause and the perfect process restoration to the in-control condition, i.e. \( C_i = 0 \) and \( Y_i = 0 \). Therefore, \( p_{m,v} = p_{0,v} \) and \( q_{m,v} = q_{0,v} \).

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