Abstract: We consider the problem of investment and consumption with a hidden Markov model and a regime switching structure. The Bayesian approach is followed to integrate econometric consideration and to make inference of the hidden Markov model. The optimal investment strategy is characterized by the method of stochastic dynamic programming and simulation results are given.

Keywords: Bayesian inference, investment and consumption, Markov Chain Monte Carlo (MCMC), portfolio optimization, regime switching, stochastic dynamic programming (Markov decision processes).

1. Introduction

According to Korn and Korn [11], the problem of portfolio selection (also known as investment and consumption) deals with the situation in which an investor must determine how many shares of which assets to hold at which time instants and how much wealth to consume in order to maximize the expected total utility from all the consumptions over the entire investment horizon, or from the terminal wealth at the end of investment. Due to significant risks in the investment returns, the optimal choice between investment and consumption often means sacrificing some current welfare for better understanding the financial market consisting of risky assets and gaining possibly higher wealths in the future. Therefore the principle of investment and consumption is to search for good trade-off between investment returns and risks.

Markowitz [12] pioneered the modern study of portfolio selection, followed by important contributions by Samuelson [19], Merton [14-15], and Black and Scholes [5]. The method of stochastic dynamic programming (also known as Markov decision processes, see Puterman [17]) has been traditionally and still remains to be the most powerful technique to solve problems of portfolio selection, for discrete-time multi-period and continuous-time models. For example, Samuelson [19] applied stochastic dynamic programming to maximize the overall expected utility of consumption for a discrete-time, multi-period investment and consumption model. Merton [14-15] extended the model to a continuous-time setup and used stochastic dynamic programming to obtain the optimal portfolio strategy for a continuous-time Brownian-motion model. More recent works on applying stochastic dynamic programming in mathematical finance include Bielecki et al. [4], Schäl [20] and Bäuerle and Rieder [2].

In general, the process of investment and consumption normally consists of five steps according to Sharpe et al. [21]. In the first step, the investor assesses his/her investment
objective, risk tolerance level and the amount of investable wealth. In the second step, the investor performs a financial analysis on the available investment assets and the principal goal is to identify the expected returns and corresponding risks of different assets. In the third step, the portfolio is constructed which involves identifying the specific assets for the investment portfolio and allocating the proportions of investable wealth among the selected assets. The fourth step is called portfolio revision and involves the periodic repetition of the first three steps. In this step, the investor periodically modifies the investment objectives and adjusts the portfolio according to the changing situation of the financial market. Transaction costs may incur during this step. In the fifth step, the investor evaluates periodically the portfolio performance and makes necessary adjustments, in terms of both the investment returns earned and the risks experienced.

Bayesian learning has been applied to portfolio selection problems, originated with Zellner and Chetty [28] and extended in Bawa et al. [3], Epstein and Miao [9] and Xia [26]. The Bayesian method has also been incorporated with the method of stochastic dynamic programming, see for example Van Hee [22] and Martin [13]. Market conditions have been recently incorporated in portfolio selection problems using regime switching (or Markov modulated) models, in which key financial parameters (like asset return rates and volatility rates, and hence the investment returns) depend on the market regime (or financial mode) of the economic environment. A (discrete-time or continuous-time) Markov process modulates the stochastic switch of the regime. See Cheung and Yang [6-7] and Bäuerle and Rieder [1] for portfolio selection problems with discrete-time regime switching models and Yin and Zhou [27] and Zhou and Yin [29] for portfolio selection problems with continuous-time regime switching models. Elliott and Hinz [8] considered portfolio optimization using a hidden Markov model in the context of a Black-Scholes type financial market.

In our recent work Wang and Yi [25], we examined portfolio selection problems in which the relative return rate from the risky asset is stochastic with an unknown parameter in its distribution. Using the Bayesian approach and stochastic dynamic programming, we incorporated the concept of risk aversion into the model and characterized the optimal strategies for both the power and logarithmic utility functions with a constant relative risk aversion. Numerical results supported the intuition that a higher proportion of investment should be allocated to the risky asset if the mean return rate on the risky asset is higher and/or the risky asset return rate is less volatile. Then in Wang and Wang [24], we applied the statistical model of bandit processes to formulate and solve two kinds of portfolio problems in which the dividend payment frequency is stochastic following a Poisson distribution with unknown intensity rates.

In this paper, we consider the problem of portfolio selection with a hidden Markov model and a regime switching structure. In particular, we assume that the investment return rate from the risky asset depends on the unknown state of the hidden Markov model and the transition probabilities of the regime switching model are unknown. The Bayesian approach is followed to integrate econometric considerations into the model. To meet the significant challenge of high dimensional integration, we use the discretization-based sampling scheme introduced in Fu and Wang [10] and Wang and Fu [23].

The paper is organized as follows. In Section 2, we introduce the problem of portfolio selection with a hidden Markov model and a regime switching structure. Main simulation results are presented in Section 3. Section 4 concludes the paper.
2. Portfolio Selection with a Hidden Markov Model

We incorporate a hidden Markov model with a regime switching structure in the problem of portfolio selection. The hidden Markov model is considered to be a bivariate stochastic process, one of which is unobservable and constitutes a Markov chain, while the other is observable and offers information about the hidden states.

For the purpose of illustration, we examine a financial market consisting of one risky asset and one riskless asset. The portfolio (i.e., investment and consumption) decisions are made at time points \( n = 1, 2, ..., N \). The riskless asset offers a constant interest rate \( r \) over each single time period. However, the return rate made at time points \( n \) of a hidden Markov model \( \{ X_n, n = 1, 2, ..., N \} \), which represents a stochastic process of the economic environment.

For simplicity, we assume that the unobservable hidden Markov model \( \{ X_n, n = 1, 2, ..., N \} \) is an irreducible time-homogeneous Markov chain with a state space \( S \) and transition probability matrix \( P \). To incorporate a regime switching structure, we further assume that \( S = \{ 1, 2 \} \), where state 2 represents a favorable economic factor during the current investment period, and state 1 represents an unfavorable economics factor. A favorable economic factor means that the economic environment has a positive impact on the stock price during the current investment period, while an unfavorable economic factor means a negative impact. The transition probability matrix \( P = (p_{ij}), i, j = 1, 2 \), describes the changing economic environment.

Within this framework, we assume that the distribution of \( \{ R_n \} \) given the hidden state \( \{ X_n \} \) belongs to a specified parametric family so that \( R_n \mid (X_n = i) \sim f(r_i \mid \theta_i) \), where \( f \) is a time-homogeneous probability density function or a probability mass function and \( \theta_i \) is an unknown parameter, \( i = 1, 2 \). At each time \( n = 2, ..., N \), using the observed risky asset return rates \( r_1, ..., r_{n-1} \) from the past investment periods, we can estimate the current unobservable hidden states \( X_n \), the transition probabilities \( P \) and the unknown parameters \( \theta_i, i = 1, 2 \). We assume that the economic environment is not subject to dramatic changes. We then use the estimated results to predict the risky asset return rate and make a portfolio decision.

Starting with a given initial wealth of \( W_0 = w \) and a given initial state \( X_0 = i \in S \) at time 1 (or a given distribution of the initial state), the investor consumes amount \( c_1 \), \( 0 \leq c_1 < w \) and invests \( W_1 = w - c_1 \) in the financial market. Of the amount \( W_1 \), a proportion of \( \phi_1 \in [0, 1] \) is invested in the risky asset and the remaining proportion \( 1 - \phi_1 \) is invested in the riskless asset. In general, the total wealth of the investor at time \( n \) is denoted as \( W_n \) and the consumption during the time period \( n \) is denoted as \( c_n \), where \( 0 \leq c_n \leq W_n \). At time \( n \), the investor makes a decision about the proportion of the remaining wealth \( W_n - c_n \) to be invested in the risky asset, which is denoted as \( \phi_n \in [0, 1] \). The remaining proportion of \( 1 - \phi_n \) is invested in the riskless asset. Here, no short-selling and leveraging are allowed.

The wealth satisfies the budget constrains equation \( W_n = (W_{n-1} - c_n)(1 - \phi_n)R_n + \phi_nR_n^{s-1}, n = 1, ..., N \) and a strategy is a sequence \( \pi = \{(c_n, \phi_n), 0 \leq c_n \leq w_n, 0 \leq \phi_n \leq 1, n = 1, ..., N \} \) of investment and consumption decisions satisfying the budget constraint equation. Here, \( W_n \) is the wealth at time \( n \) and \( R_n^{s-1} \) is the value of the return rate \( R_n \) given that \( X_n = x_n \).

Given any initial state and initial wealth, the investor’s objective is to find an optimal strategy \( \pi^* = \{(c_1, \phi_1), ..., (c_N, \phi_N)\} \) to achieve
subject to the state $W_n = (W_{n-1} - c_n)[R + \varphi_n(R_{n+1} - R)]$ where $0 < \alpha < 1$ is a discounting factor, $\mu(c)$ is a measurable one-stage utility function and $B(w)$ is a measurable terminal utility function. Intuitively, the goal is to maximize the expected value of the total of discounted utility from all investment and consumption decisions subject to a self-financing process. This is typically solved by the method of stochastic dynamic programming. However in the present setting, the solution is intractable.

As indicated in Elliott and Hinz [8], a continuous function $\mu(c)$ of $c \in (0, \infty)$ is called a utility function if it is strictly increasing, strictly concave, and its derivative $\mu'(c)$ is strictly decreasing with $\lim_{x \to 0} \mu'(c) = +\infty$ and $\lim_{x \to \infty} \mu'(c) = 0$. Two utility functions are commonly considered in the literature of economics and finance: the isoelastic power function $c^{1-\gamma}/(1-\gamma)$, $0 < \gamma < 1$, $c > 0$, and the logarithmic function $\ln(c)$, $c > 0$, which can be regarded as a limiting case when $\gamma$ tends towards unity. Both have a positive first derivative and a negative second derivative, reflecting a desire for more consumption and decreasing marginal value of additional consumption. Pratt [16] defined the absolute risk aversion at point $c$ as $r(c) = -\mu''(c)/\mu'(c)$.

He showed that a decision maker A has a greater absolute risk aversion than B at all $c$ if and only if A is globally more risk aversion in the sense that A’s risk premium is always larger. Since the risk may be proportional to the size of the wealth, Pratt defined the relative risk aversion as $-\mu''(c)/\mu'(c)$ and obtained similar results. Both the isoelastic power and logarithmic utility functions have decreasing absolute risk aversions but a constant relative risk aversion (CRRA).

3. Some Simulation Results

We apply the Bayesian inference for the hidden Markov model. Similar to the ideas in Ryden et al. [18], the daily return of the risky asset depends on the unobserved economic environment by a normal distribution with mean zero and unknown variance. That is, $R_n | X_{i} = i \sim N(0, \sigma_i^2)$. We assume that the prior distribution of the unknown variance is the inverse gamma distribution suggested by Wang and Fu [23] and given as

$$\sigma_i^2 \sim IGamma \left( 2, \left( \frac{R_i}{\theta} \right)^2 \right) = IGamma(\kappa, \beta), \quad i = 1, 2, \sigma_1 > \sigma_2,$$

where $R_i$ is the range of the data set. We impose a soft lower bound to keep $\sigma_i^2$ away from being too close to zero. The density function of $\sigma_i^2$ is $f(x) = \beta^\kappa x^{\kappa-1} e^{-\beta/x} / \Gamma(\kappa)$, where $\Gamma(\kappa)$ is the gamma function. The mean of the inverse gamma distribution is $E(\sigma_i^2) = \beta / (\kappa - 1)$ and the variance is $Var(\sigma_i^2) = \beta^2 / [(\kappa - 1)^2(\kappa - 2)]$, $\kappa > 2$.

The prior distribution $A$ of the transition probability matrix is assumed to be a Dirichlet distribution. For the Dirichlet prior, we assume that $\alpha_i = 1$ for the simplest case. So $a_{ij}$, $i = 1, 2$, are uniformly distributed in the simplex. These prior distributions are considered uniform over the two-dimensional simplex and non-informative.

Since the optimal investment and consumption problem is characterized by the hidden Markov model, we simulate and compare different investment strategies. The number of states in a hidden Markov model is an important parameter that has critical impact on the model. However, the EM algorithms for likelihood inference and MCMC procedures for Bayesian analysis have various difficulties in dealing with mixture of distributions with an
unknown number of components. The Bayesian approach and the discretization-based sampling method developed by Fu and Wang [10] and Wang and Fu [23] are used to make inference of the hidden Markov model and to estimate the posterior. Based on the observed daily risky asset return rate, the posterior distributions of the model parameters were updated daily, and the updated posterior information is used to better understand the model. The expected utility under the investment and consumption strategy is simulated and estimated.

To sample from the posterior distribution, we initialize the intervals for the parameters with \( a_{11}, a_{21} \in (0,1) \) and \( \sigma_1^2, \sigma_2^2 \in (0,9.0 \times 10^{-4}), \quad \sigma_1^2 > \sigma_2^2 \). Following the discretization-based sampling, we apply the sampling scheme in four steps. Step 1. Discretization: The initial compact interval is set for the parameters with the requirement that \( \theta_1 < \theta_2 \). Step 2. Contourization: Generate 2\( \times \)10\(^5\) samples in the compact intervals to discretize the interface, and evaluate the log likelihood function \( \log \pi_i \). All the sample points are partitioned into 200 contours, with 1000 points inside each contour. All the points generated are rearranged in the descending order such that \( f(x_i) \geq x_j \) if \( i < j \). For each contour, we get \( E = \{ x_j : (i-1)l < j \leq il \}, \quad i = 1,2,\ldots,200 \), where \( l = 1000 \) is the number of sample points within each contour. Then, the union of all these contour sets is the sample space and these sets are mutually exclusive. Step 3. Sampling: A random sample of size 1000 is drawn by the first sampling contour (with replacement) from the probability mass function obtained in step 2 and then by drawing points from within each contour with equal probability. Step 4. Visualization: The sample obtained in step 3 is plotted by marginal histograms.

Simulation results are summarized as follows. Here we use the logarithmic utility function \( \ln(C) \) which can be regarded as a limiting case of the isoelastic power utility function. This logarithmic utility function has a positive first derivative and a negative second derivative, reflecting a desire for more consumption and decreasing marginal value of additional consumption. Moreover, it has a decreasing absolute risk aversion but a constant relative risk aversion. Because the model is mathematically intractable and the simulations are time consuming, we only simulate investment and consumption decisions over 5 days. The mean and standard deviation of the simulated utility are calculated. All simulations assume a 10% discount factor. All simulations were based on 1000 replications.

Table 1 shows simulated mean and standard deviation of the total expected utility for different settings of riskless asset return rates \( r \) and different proportions of investment in the risky asset \( \varphi \). Here we assume a stationary strategy in that the same proportion \( \varphi \) applies to all days. A common 10% consumption rate is assumed. Table 2 shows simulated results when the \( \kappa \) parameter of the inverse gamma prior distribution is set at \( \kappa = 2 \) but the \( \beta \) parameter varies. Again, we assume a common 10% consumption rate and a common 50% investment in the risky asset.

<table>
<thead>
<tr>
<th>interest rate</th>
<th>( r )</th>
<th>( \varphi = 0.1 )</th>
<th>( \varphi = 0.3 )</th>
<th>( \varphi = 0.5 )</th>
<th>( \varphi = 0.7 )</th>
<th>( \varphi = 0.9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>51.12 (0.16)</td>
<td>53.35 (0.60)</td>
<td>53.01 (0.04)</td>
<td>53.75 (0.09)</td>
<td>55.97 (0.02)</td>
<td></td>
</tr>
<tr>
<td>0.02</td>
<td>52.26 (0.14)</td>
<td>54.54 (0.43)</td>
<td>53.53 (0.02)</td>
<td>55.63 (0.03)</td>
<td>56.14 (0.01)</td>
<td></td>
</tr>
<tr>
<td>0.03</td>
<td>54.99 (0.34)</td>
<td>56.56 (0.47)</td>
<td>54.90 (0.01)</td>
<td>56.86 (0.24)</td>
<td>57.16 (0.18)</td>
<td></td>
</tr>
<tr>
<td>0.04</td>
<td>55.52 (0.26)</td>
<td>57.14 (0.69)</td>
<td>55.85 (0.01)</td>
<td>58.46 (0.10)</td>
<td>57.91 (0.03)</td>
<td></td>
</tr>
</tbody>
</table>
Finally, Table 3 shows simulated mean and standard deviation of the total expected utility for different settings of riskless asset return rates $r$ and different prior distribution for the transition probabilities. Here we assume a stationary strategy in that the same proportion $\varphi = 0.5$ applies to all days. A common 10% consumption rate and a common 50% investment in the risky asset are assumed.

### Table 3. Total utility for 5 days, mean (standard deviation) with different interest rate $r$ and different prior for the transition probabilities.

<table>
<thead>
<tr>
<th>Interest rate $r$</th>
<th>$(0.9,0.1)$</th>
<th>$(0.7,0.3)$</th>
<th>$(0.5,0.5)$</th>
<th>$(0.3,0.7)$</th>
<th>$(0.1,0.9)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>59.47 (0.08)</td>
<td>61.02 (0.36)</td>
<td>62.04 (0.38)</td>
<td>71.07 (2.24)</td>
<td>66.02 (1.05)</td>
</tr>
<tr>
<td>0.02</td>
<td>59.51 (0.11)</td>
<td>61.98 (0.02)</td>
<td>63.86 (0.01)</td>
<td>71.73 (0.41)</td>
<td>66.90 (0.91)</td>
</tr>
<tr>
<td>0.03</td>
<td>61.65 (0.12)</td>
<td>64.09 (0.25)</td>
<td>69.12 (0.50)</td>
<td>71.82 (0.86)</td>
<td>71.54 (0.94)</td>
</tr>
<tr>
<td>0.04</td>
<td>63.65 (0.11)</td>
<td>64.88 (0.30)</td>
<td>69.19 (0.11)</td>
<td>79.56 (2.09)</td>
<td>76.26 (0.16)</td>
</tr>
</tbody>
</table>

### 4. Conclusions

In this paper, we formulate the investment and consumption problem when the financial market is modulated by a hidden Markov model with a regime switching structure. We assume no bankruptcy and a finite investment horizon. We estimate the parameters of the hidden Markov model and then make portfolio decisions for the investment and consumption problem based on the Bayesian inference of the hidden Markov model. Among the available methods for parameter estimation, we apply the discretization-based sampling method to handle high-dimension integral problems. We characterize the optimal investment strategy by the method of stochastic dynamic programming.

We consider the effect of various choices of the model parameters such as the constant interest rate from the riskless asset, different proportions of consumption and different prior distributions. Although incorporating a hidden Markov model and a regime switching structure makes the portfolio selection problems more realistic, the mathematical intractability and computational complexity make the model infeasible for explicit solutions. The practical challenge is difficult to overcome. Due to this, we are only able to simulate several possibilities of the investment and consumption strategies and achieve limited findings.

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References


Authors’ Biographies:

Yi Hua received her M.Math. degree from Lakehead University, Canada and M.Sc. degree in Statistics from University of Manitoba, Canada in 2006 and 2008, respectively. From early 2008 to 2010, she joined National Research Council of Canada as a research assistant. Her research areas include hidden Markov modeling, adaptive designs, stochastic modeling and mathematical computations. She has been a Ph.D. student in the Department of Statistics, University of Manitoba, since September 2012 and a Biostatistician at Cangene Corporation since the summer of 2008.

Xikui Wang is Professor and Head, in the Department of Statistics, University of Manitoba, Canada. He received his M.Sc. in Mathematics and Ph.D. in Statistics, both from the Department of Mathematics and Statistics, University of Saskatchewan, Canada. His research interests include Markov decision processes, bandit processes, response adaptive design of clinical trials and mathematics finance.