Joint Reliability Importance in a Binary $k$-out-of-$n$: G System with Exchangeable Dependent Components

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Abstract: In this paper, we study joint reliability importance (JRI) in a $k$-out-of-$n$: G structure consisting of exchangeable dependent components. We obtain a closed-form formula for the JRI of multiple components of a $k$-out-of-$n$: G system with dependent components. We illustrate the results for the $k$-out-of-$n$: G model under stress-strength setup. The results extend and generalize the results in the literature from various perspectives including exchangeable type dependence for the JRI of two components.

Keywords: Exchangeability, joint reliability importance, $k$-out-of-$n$: G system, stress-strength reliability.

1. Introduction

Component importance measures are of special interest in reliability due to their crucial role in optimal design, and improvement of systems. The concept of component importance (marginal) was first introduced by Birnbaum [2]. Marginal reliability importance measures the change in the system reliability with respect to the change in reliability of a specific component. Different kind of marginal importance measures have been studied in Xie and Bergman [13], Boland and Neweih [3]. For recent reviews of the topic, see Elsayed [4] and Kuo and Zhu [11]. On the other hand, joint reliability importance (JRI) of two components is a measure of the interaction of two components in a system for their contribution to the system reliability (Hong and Lie [9], Hagstrom [8], Armstrong [1]). Let $R(p)$ denote the reliability of a system consisting of $n$ components with reliabilities $p_1,...,p_n$, where $p=(p_1,...,p_n)$. Then, the marginal Birnbaum [2] importance of the $i$th component is defined as

$$MRI(c_i) = \frac{\partial R(p)}{\partial p_i},$$

for $i=1,...,n$. Motivated by Birnbaum [2] marginal reliability importance, the JRI of two components is defined by

$$JRI(c_i,c_j) = \frac{\partial^2 R(p)}{\partial p_i \partial p_j},$$

for $i \neq j$, and $i,j=1,...,n$. Hagstrom [8] introduced the concepts of "reliability substitutes" and "reliability complements" depending on the sign of JRI of two components.

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To make these concepts clearer, consider the system depicted in Figure 1 which consists of independent components with reliabilities $p_1$, $p_2$, and $p_3$.

![Figure 1. Three component system.](image)

The reliability of the entire system is obtained as

$$R(p) = p_1 p_2 + p_1 p_3 - p_1 p_2 p_3.$$  

Therefore

$$JRI(c_1, c_2) = 1 - p_3 > 0, \quad JRI(c_1, c_3) = 1 - p_2 > 0, \quad JRI(c_2, c_3) = -p_1 < 0,$$

which implies that the components "$c_1" and "$c_2" and "$c_1" and "$c_3" are reliability complements, and "$c_2" and "$c_3" are reliability substitutes.

Hong et al. [10] studied JRI of components in a $k$-out-of-$n:G$ system. In particular, they have presented a closed-form equation for the JRI of two components, and investigated its properties with respect to component reliabilities, and system parameters $k$ and $n$. The results in Hong et al. [10] are mostly based on the assumption of independent components. Gao et al. [6] presented a detailed analysis for JRI of three components in a $k$-out-of-$n:G$ system with independent components. They have also studied various conditional reliability importance measures. Rani et al. [12] studied conditional marginal and conditional joint reliability importance in series-parallel systems.

In the present paper, we extend and generalize some of the results in Hong et al. [10] and Gao et al. [6] to the $k$-out-of-$n:G$ system consisting of exchangeable dependent components. The paper is organized as follows: Section 2 gives a closed-form formula for the JRI of multiple components of a $k$-out-of-$n:G$ system with dependent components. Some properties of JRI of two components in exchangeable dependent case are also presented. These results are illustrated for the $k$-out-of-$n:G$ model under stress-strength setup.

2. JRI for Dependent Components

Consider a binary coherent system consisting of components. Let $X_i$ denote the state of the $i^{th}$ component, where $X_i = 1$ if the $i^{th}$ component works, and $X_i = 0$, otherwise, $i = 1, 2, \ldots, n$. Let $E$ be the event that the system works. Then for arbitrarily dependent components, the joint reliability importance of two and three components may be defined respectively as

$$JRI(c_1, c_2) = P\{E|X_{c_1} = 1, X_{c_2} = 1\} - P\{E|X_{c_1} = 1, X_{c_2} = 0\} - P\{E|X_{c_1} = 0, X_{c_2} = 1\} + P\{E|X_{c_1} = 0, X_{c_2} = 0\},$$

and
JRI\( (c_1, c_2, c_3) = P\left\{ E \left| X_{c_1} = 1, X_{c_2} = 1, X_{c_3} = 1 \right\} - P\left\{ E \left| X_{c_1} = 1, X_{c_2} = 1, X_{c_3} = 0 \right\} \right\}
\quad - P\left\{ E \left| X_{c_1} = 1, X_{c_2} = 0, X_{c_3} = 1 \right\} - P\left\{ E \left| X_{c_1} = 0, X_{c_2} = 1, X_{c_3} = 1 \right\} \right\}
\quad + P\left\{ E \left| X_{c_1} = 1, X_{c_2} = 0, X_{c_3} = 0 \right\} + P\left\{ E \left| X_{c_1} = 0, X_{c_2} = 1, X_{c_3} = 0 \right\} \right\}
\quad + P\left\{ E \left| X_{c_1} = 0, X_{c_2} = 0, X_{c_3} = 1 \right\} - P\left\{ E \left| X_{c_1} = 0, X_{c_2} = 0, X_{c_3} = 0 \right\} \right\}.

The joint reliability importance can be generalized to \( m \) components easily and the result is given in the following Lemma.

**Lemma 1.** For a system consisting of arbitrarily dependent components, the joint reliability importance of \( m \) components is

\[
\text{JRI}(c_1, \ldots, c_m) = \sum_{j=0}^{m} (-1)^j \sum_{i_1, \ldots, i_j \in \{1, \ldots, m\}, i_1 < \ldots < i_j} P\left\{ E \left| X_{i_1} = 0, \ldots, X_{i_j} = 0, X_{i_j+1} = 1, \ldots, X_m = 1, \right\} 1 \right\},
\]

where \( C_m = \{1, \ldots, m\} \), and

\[
D_{m,j} = \{(i_1, \ldots, i_j) : i_1, \ldots, i_j \in \{1, \ldots, m\}, i_1 < \ldots < i_j, j = 1, \ldots, m\}.
\]

Let us assume that the states of the components are exchangeable, i.e. the joint distribution of \( X_1, \ldots, X_n \) is invariant under permutation of its arguments. That is, for each \( n > 0 \),

\[
P\left\{ X_{a_1} = x_1, \ldots, X_{a_n} = x_n \right\} = P\left\{ X_{j_1} = x_1, \ldots, X_{j_n} = x_n \right\},
\]

for any permutation \( (\pi_1, \ldots, \pi_n) \) of the indices in \( \{1, \ldots, n\} \). Under exchangeability, any sequence with \( a \) 1s and \( n-a \) 0s, has probability

\[
g(n,a) = P\left\{ X_1 = 1, \ldots, X_a = 1, X_{a+1} = 0, \ldots, X_n = 0 \right\} = \sum_{i=0}^{n-a} (-1)^i \binom{n-a}{i} \lambda_{a+i} = \sum_{i=0}^{a} (-1)^i \binom{a}{i} \theta_{a-i},
\]

where \( \lambda_a = P\{X_1 = 1, \ldots, X_a = 1\} \), and \( \theta_a = P\{X_1 = 0, \ldots, X_a = 0\} \) with \( \lambda_0 = 1 \), \( \theta_0 = 1 \) (see, e.g. George and Bowman [7]).

In the following, we study the JRI of components in a \( k \)-out-of-\( n \): G system with exchangeable components. Before we proceed, we note that the reliability of a \( k \)-out-of-\( n \): G system consisting of exchangeable dependent components is computed from

\[
P\{S_n \geq k\} = \sum_{j=k}^{n} g(n,j),
\]

where \( S_n = \sum_{i=1}^{n} X_i \) is the number of working components, and \( g(n,j) \) is defined by (1).

In the following Theorem, we obtain the joint reliability importance of \( m \) components in a \( k \)-out-of-\( n \): G system with exchangeable components. All proofs are presented in the Appendix.

**Theorem 1.** For a \( k \)-out-of-\( n \): G system consisting of exchangeable dependent components, the joint reliability importance of \( m \) components is

\[
\text{JRI}(c_1, \ldots, c_m) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \frac{1}{g(m,m-j)} \sum_{a=m+j}^{n-m} \binom{n-m}{a-m+j} g(n,a),
\]

for \( k \geq m \).
The following Corollary is immediate from Theorem 1 for $m=2$, and it extends Theorem 1 of Hong et al. [10] to exchangeable dependent components.

**Corollary 1.** For a $k$-out-of-$n$:G system consisting of exchangeable dependent components, the joint reliability importance of $m=2$ components is

$$
\text{JRI}(c_1, c_2) = \sum_{m=k}^{n} \frac{1}{\lambda_2} \binom{n-2}{m-2} g(n, m) - \frac{2}{\lambda_1 - \lambda_2} \sum_{m=k}^{n-1} \binom{n-2}{m-1} g(n, m, m) + \frac{1}{1 - 2\lambda_1 + \lambda_2} \sum_{m=k}^{n-2} \binom{n-2}{m} g(n, m),
$$

for $k \geq 2$, where $\lambda_2 = P\{X_1 = \ldots, X_k = 1\}$.

**Remark 1.** An equivalent representation for JRI$(c_1, c_2)$ is given by

$$
\text{JRI}(c_1, c_2) = \frac{2}{\theta_1 - \theta_2} \sum_{m=1}^{k-1} \binom{n-2}{m-1} g(n, m) - \frac{1}{\theta_2} \sum_{m=0}^{k-1} \binom{n-2}{m} g(n, m) - \frac{1}{1 - 2\theta_1 + \theta_2} \sum_{m=2}^{n-2} \binom{n-2}{m-2} g(n, m).
$$

The following result can be immediately obtained from Theorem 1 taking $\lambda_2 = p^*$. It extends Theorem 1 of Hong et al. [10] to $m$ components.

**Corollary 2.** For a $k$-out-of-$n$:G system consisting of i.i.d. components with common component reliability $p$, the joint reliability importance of $m$ components is

$$
\text{JRI}(c_1, \ldots, c_m) = \min(m, n-k) (-1)^j \sum_{j=0}^{\min(m, n-k)} \binom{m}{j} \binom{n-m}{a-m+j} p^{a+m+j} (1-p)^{n-a-j},
$$

for $k \geq m$.

Below, we perform a detailed analysis for 2-out-of-$n$:G systems, and some of the results in Hong et al. [10] are generalized to exchangeable dependent case. First, for $k=2$ in Remark 1 we have

$$
\text{JRI}(c_1, c_2) = \frac{2(\theta_{a-1} - \theta_a)}{\theta_1 - \theta_2} - \frac{\theta_a + (n-2)(\theta_{n-1} - \theta_n)}{\theta_2}.
$$

(2)

**Theorem 2.** Let JRI$_n(c_1, c_2)$ denote the joint reliability importance of components in 2-out-of-$n$:G system consisting of exchangeable dependent components $(n \geq 3)$. Then JRI$_n(c_1, c_2) < \text{JRI}_{n+1}(c_1, c_2)$ when $A = \theta_{a-1} - 2\theta_a + \theta_{a+1} > 0$, and

$$
\frac{\theta_a}{\theta_1} < 1 - \frac{2}{n}.
$$

(3)

An interesting open problem is the determination of conditions based on $\theta_a$s for a $k$-out-of-$n$:G system such that JRI$_n(c_1, c_2) < \text{JRI}_{n+1}(c_1, c_2)$.

The proof of the following result is immediate from Theorem 2 since for i.i.d. components

$$
A = \theta_{a-1} - 2\theta_a + \theta_{a+1} = q^{a-1} - 2q^a + q^{a+1} = q^{a-1}(1-q)^2 > 0.
$$

**Corollary 3.** For a 2-out-of-$n$:G system consisting of i.i.d. components with common component reliability $p = 1 - q$, JRI$_n(c_1, c_2) < \text{JRI}_{n+1}(c_1, c_2)$ when $q < 1 - 2/n$. 

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2.1. Application in Stress-Strength Reliability

Consider a $k$-out-of-$n : G$ system whose components are subjected to a common random stress $Z$. Let $Y_1, \ldots, Y_n$ denote the random strengths of the components. The component fails if the applied stress exceeds its strength at any moment, i.e. if $Y_i > Z$ then the $i^{th}$ component operates, otherwise fails. The probability $P\{Y_i > Z\}$ defines the reliability of the $i^{th}$ component. The binary states of components are defined as

$$X_i = \begin{cases} 1, & \text{if } Y_i > Z, \\ 0, & \text{if } Y_i \leq Z, \end{cases}$$

for $i = 1, \ldots, n$, where we assume that $Y_1, \ldots, Y_n$ are independent random variables having common continuous cumulative distribution function (cdf) $F(x) = P\{Y_i \leq x\}$, $i = 1, \ldots, n$ and independent of the random stress $Z$ having continuous cdf $G(x) = P\{Z \leq x\}$. It is clear that the states of the components $X_1, \ldots, X_n$ are exchangeable dependent with

$$\lambda_a = P\{X_1 = 1, \ldots, X_a = 1\} = \int_0^{\infty} (1 - F(z))^a dG(z),$$

and

$$\theta_a = P\{X_1 = 0, \ldots, X_a = 0\} = \int_0^{\infty} F^a(z) dG(z). \hspace{1cm} (4)$$

Under this setup, for given $F$ and $G$, the reliability of the $k$-out-of-$n : G$ system is calculated from

$$P\left\{\sum_{i=1}^{n} X_i \geq k\right\} = \sum_{j=k}^{n} g(n,j),$$

where

$$g(n,a) = \sum_{i=0}^{a} (-1)^i \binom{n-a}{i} \lambda_{a+i} = \sum_{i=0}^{a} (-1)^i \binom{a}{i} \theta_{a+i}.$$ 

The reliability of an arbitrary coherent system under a stress-strength setup is studied in Eryilmaz [4].

For an illustration of the results presented in the previous sections, let $F(x) = 1 - e^{-\alpha x}$, and $G(x) = 1 - e^{-\beta x}$, for $x > 0$. Then from Equation (4), we obtain

$$\theta_a = \int_0^{\infty} (1 - e^{-\alpha x})^a \beta e^{-\beta x} d\beta = \sum_{s=0}^{a} (-1)^s \binom{a}{s} \frac{1}{1 + \rho s},$$

for $a \geq 1$, where $\rho = \alpha / \beta$. In particular, we have

$$\theta_1 = P\{Y_1 < Z\} = \frac{\alpha}{\alpha + \beta} = \frac{1}{1 + \frac{1}{\rho}},$$

and

$$\theta_2 = P\{Y_1 < Z, Y_2 < Z\} = \frac{2\alpha}{2\alpha + \beta} = \frac{1}{1 + \frac{1}{2\rho}}.$$

In Figure 2, we plot $\text{JRI}_n(c_1, c_2)$ for 2-out-of-$n : G$ systems when $n = 3, 4$ and $5$ as a function of $\rho$ under the above mentioned stress-strength model. From Theorem 2, we obtain that $\text{JRI}_n(c_1, c_2) < \text{JRI}_{n+1}(c_1, c_2)$ when $\theta_2 / \theta_1 < 1 - 2 / n$ which implies $\rho < (n-2) / 4$, for $n > 2$. 
This can also be numerically observed from Figure 2. For example, $\text{JRI}_3(c_1,c_2) < \text{JRI}_4(c_1,c_2)$ when $\rho < 0.25$.

Table 1. The sign of $\text{JRI}_n(c_1,c_2)$ when $\rho = 0.5$ and $\rho = 0.75$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$k$</th>
<th>$\rho = 0.5$</th>
<th>$\rho = 0.75$</th>
<th>$n$</th>
<th>$k$</th>
<th>$\rho = 0.5$</th>
<th>$\rho = 0.75$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>5</td>
<td>-</td>
<td>-</td>
<td>100</td>
<td>5</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>10</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>20</td>
<td>+</td>
<td>-</td>
<td>30</td>
</tr>
<tr>
<td>50</td>
<td>5</td>
<td>+</td>
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<td>+</td>
<td>-</td>
</tr>
<tr>
<td>30</td>
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<td>10</td>
<td>40</td>
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</table>

Table 1 shows the sign of $\text{JRI}_n(c_1,c_2)$ for large values of $n$ and $k$ when $\rho = 0.5$ and $\rho = 0.75$ in the above-mentioned stress-strength model. From the Table we observe that the sign of $\text{JRI}_n(c_1,c_2)$ depends not only on the system parameters $k$ and $n$ but also the statistical model parameter $\rho$.

Additionally, let $Y \sim N(\mu_Y, \sigma_Y^2)$ and $Z \sim N(\mu_Z, \sigma_Z^2)$. In Table 2, we compute $n^*$, minimum value of $n$, such that $\text{JRI}_n(c_1,c_2) < \text{JRI}_{n+1}(c_1,c_2)$ using Theorem 2 for various values of the parameters $\mu_Y, \sigma_Y, \mu_Z, \sigma_Z$.

Table 2. Minimum value of $n$ such that $\text{JRI}_n(c_1,c_2) < \text{JRI}_{n+1}(c_1,c_2)$.

<table>
<thead>
<tr>
<th>$\mu_Y$</th>
<th>$\sigma_Y$</th>
<th>$\mu_Z$</th>
<th>$\sigma_Z$</th>
<th>$n^*$</th>
</tr>
</thead>
<tbody>
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<td>2</td>
<td>1</td>
<td>4</td>
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<td>3</td>
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<tr>
<td>3</td>
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<td>3</td>
<td>2</td>
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</tr>
<tr>
<td>4</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

A $k$-out-of-$n$:G system which is defined in a stress-strength setup might be useful in various real life situations. An airplane which is capable of functioning if and only if at least two of its three engines are functioning can be modeled by a $2$-out-of-$3$:G system. The
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 engines of the airplane operate under various stresses, and if the stress exceeds the strength of the engine then the engine fails. For an illustration, let \( Y \sim N(\mu_Y, \sigma_Y^2) \) and \( Z \sim N(\mu_Z, \sigma_Z^2) \). That is, the strength and stress distributions are both normal. Then for \( \mu_Y = 10, \sigma_Y = 1, \mu_Z = 3, \sigma_Z = 2 \), the JRI between two engines in 2-out-of-3:G system is \( \text{JRI}(c_1, c_2) = -0.7168 < 0 \) which indicates that one engine becomes less important when the other is functioning due to the definition in Armstrong [1].

3. Summary and Conclusions

In this paper, we studied the joint reliability importance in a binary \( k \)-out-of-\( n \) system that consists of exchangeable dependent components. Although the JRI for such a system with independent components has been examined in several papers, the case of exchangeable dependence has not been investigated. In Lemma 1, we obtained an expression for the JRI of \( m \) components for the most general case when the components are dependent, and nonidentically distributed.

The novelty of the present paper lies in the consideration of statistical dependence for studying JRI in a \( k \)-out-of-\( n \) : G system. We have also obtained a closed-form expression for the JRI of \( m \) components in a \( k \)-out-of-\( n \) : G system with independent and identical components. This expression generalizes the result of Hong et al. [10] to \( m \) components. Although we focus on a certain dependence model in this paper, some other dependence models can also be considered as future research. A possible future work may also include the study of JRI for more general system structures.

Acknowledgements

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References

Appendix

**Proof of Theorem 1:** For a $k$-out-of-$n : G$ system consisting of exchangeable dependent components, using Lemma 1, the joint reliability importance of $m$ components is

$$\text{JRI}(c_1, \ldots, c_m) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} P\{S_n \geq k | j \text{ of } X_1, \ldots, X_m \text{ are 0 and } m-j \text{ of } X_1, \ldots, X_m \text{ are 1}\}. \quad (5)$$

It is clear that

$$P\{S_n \geq k | j \text{ of } X_1, \ldots, X_m \text{ are } "0" \text{ and } m-j \text{ of } X_1, \ldots, X_m \text{ are } "1"\}$$

$$= \sum_{a-k}^{n-j} \frac{1}{g(m,m-j)} P\{S_{n-m} = a-m+j | j \text{ of } X_1, \ldots, X_m \text{ are } "0", m-j \text{ of } X_1, \ldots, X_m \text{ are } "1"\} \quad (6)$$

$$= \sum_{a-k}^{n-j} \frac{1}{g(m,m-j)} \binom{n-m}{a-m+j} g(n,a).$$

The result follows using (6) in (5).

**Proof of Theorem 2:** From (2), it is easy to see that

$$\text{JRI}_n(c_1,c_2) - \text{JRI}_{n+1}(c_1,c_2) = A \left[ \frac{2}{\theta_1 - \theta_2} - \frac{(n-2)}{\theta_2} \right]$$

Thus if $A > 0$, the sign of $\text{JRI}_n(c_1,c_2) - \text{JRI}_{n+1}(c_1,c_2)$ is negative when

$$\left[ \frac{2}{\theta_1 - \theta_2} - \frac{(n-2)}{\theta_2} \right] < 0$$

which implies the condition (3). Thus the proof is complete.

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