Design of Equivalent Accelerated Life Testing Plans under Different Stress Applications

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Abstract: Accelerated Life Testing (ALT) is an efficient approach to obtain failure observations by subjecting the test units to stresses severer than design stresses and utilize the test data to predict reliability at normal operating conditions. ALT is usually conducted under constant-stresses which need a long time at low stress levels to yield sufficient failure data. Many stress loadings, such as ramp-stresses obtain failure times faster than constant-stresses but the accuracy of reliability predictions based on such loadings has not yet been investigated. We develop test plans under different stress applications such that the reliability prediction achieves equivalent statistical precision to that of the constant-stress. The research shows indeed there are such equivalent plans that reduce the test time, minimize the cost and result in the same accuracy of reliability predictions.

Keywords: Accelerated life testing (ALT), equivalency, maximum likelihood, proportional hazard, ramp test.

Acronyms
ALT accelerated life test.
ASVar asymptotic variance.
CE cumulative exposure.
MLE maximum likelihood estimate/estimator.
PH, PHM proportional hazard, proportional hazard model.

Notations
\( f_i() \) objective function for stress loading \( i, i = B, S, R \), where \( B, S \), and \( R \) are baseline constant-stress, step-stress and ramp-stress, respectively.
\( F_i() \) Fisher Information matrix for stress loading \( i, i = B, S, R \).
\( h_b(t) \) baseline hazard function.
\( h(t;z) \) hazard function under stress \( z \).
\( R(t;z) \) reliability function under stress \( z \).
\( H(t;z) \) cumulative hazard function under stress \( z \).
\( \Psi(t;z) \) cumulative failure time function under stress \( z \).
1. Introduction

Accelerated life testing is conducted under severer conditions than the normal operating conditions in order to obtain failure time data of test units in a much shorter time than testing at normal operating conditions. Typical ALT plans require the determination of stress types (temperature, humidity, electric field…), stress levels, allocation of test units to the stress levels and duration of the test. ALT is usually conducted under constant-stresses during the entire test duration. In practice, the constant-stress test needs a long time at low stress levels to yield sufficient failure data. This has prompted industry to consider other stress loadings (application), such as step-stress (simple or multiple), ramp-stress, sinusoidal-cyclic stress or their combinations, as shown in Figure 1.

![Various Stress Loading Types](image)

Figure 1. Various Stress Loading Types.

However, each stress loading has both advantages and drawbacks. Complicated stress profiles may yield failures in a much shorter time than constant-stress test but create challenges in the development of regression analysis models that relate stress effects to the lifetime at normal operating conditions. Thus the accuracy of the reliability prediction might
be affected. This has raised many practical questions such as: Can accelerated testing plans involving different stress loadings be designed such that they are equivalent? What are the measures of equivalency?

Literature review shows that current research on planning ALT has been focused on the design of optimum testing plans for given stress loading. For instance, the constant-stress ALT plans have been investigated in [8-11, 13-15, 20]; the step-stress ALT plans have been studied in [1, 5-7, 12, 17-19]; and the ramp-stress ALT plans have been considered in [2-4, 16]. The wide range of stress applications, stress levels and corresponding test durations give rise to the investigation of the equivalency between test plans. However, fundamental research on the equivalency of test plans has not yet been addressed in the reliability engineering field. Without understanding of such equivalency, it is difficult for practitioners to determine the best experimental settings before conducting actual ALT.

In this paper, we present definitions of equivalent test plans, propose an approach for the design of equivalent ALT plans and apply the method to the design of equivalent test plans under single constant-stress, step-stress and ramp-stress. The initial results show that it is feasible to design equivalent and yet economical and efficient ALT plans having the same accuracy of reliability prediction. We also develop a model based on the well known Cumulative Exposure (CE) assumption to investigate the life-stress relationship under general time-varying stresses, e.g. ramp-stress.

2. Definitions of Equivalent ALT Plans

In design of ALT plans, estimates of one or more reliability characteristics, such as the model parameters, hazard rate and the mean time to failure at certain conditions are common. Accordingly, different optimization criteria might be considered. For instance, if estimate of the model parameters is the main concern, D-optimality which maximizes the determinant of the Fisher information matrix is considered an appropriate criterion. When estimate of the time to quantile failure is of interest then the variance optimality that minimizes the asymptotic variance of time to quantile failure at normal operating conditions is commonly used. Meanwhile, different methods, e.g. Maximum Likelihood Estimate (MLE) or Bayesian estimator can be used for estimation of the model parameters. However each method has its inherent statistical properties and efficiencies. In light of this, we discuss equivalent test plans with respect to the same reliability characteristics and optimization criterion and determine equivalent test plans using the same inference procedure. In this paper, we propose four possible definitions of equivalency as follows:

**Definition 1:** Two test plans are equivalent if the absolute difference of the objectives for reliability prediction is less than $\delta (\delta \geq 0)$ under the same set of constraints on the number of test units, expected number of failures or total test time.

**Definition 2:** Two test plans are equivalent if they achieve the same objective for reliability prediction under the same constraints on the number of test units, expected number of failures or total test time within a margin $\delta (\delta \geq 0)$.

**Definition 3:** For the same reliability properties and inference procedure, two ALT plans are equivalent if they generate the same values of the same optimization criterion.

**Definition 4:** Two ALT plans are equivalent if the difference between the estimated times to failure and the respective confidence intervals by the plans at normal operating conditions are within $\delta (\delta \geq 0)$, where $\delta$ is an acceptable level of deviation.
3. Determining Equivalent ALT Plans

According to above definitions, the equivalent test plans are not unique. In this section, we discuss an approach for determining optimal equivalent ALT plans based on Definitions 1 and 2.

The first step of the approach is to obtain an optimal baseline test plan. Since constant-stress test is the most commonly conducted accelerated life testing in industry and its statistical inference has been extensively investigated, we propose to use an optimal constant-stress plan as a baseline.

Suppose an optimal baseline test plan can be determined from the following general formulation,

\[
\begin{align*}
\text{Min} & \quad f_B(x) \\
\text{s.t.} & \quad Lb \leq x \leq Ub, \quad C(x) \leq 0, \quad C_{eq}(x) = 0,
\end{align*}
\]

where \( f_B(x) \) is the objective function (e.g. the asymptotic variance of mean time to failure) and \( x \) is its decision variable which can be expressed as either a vector or a scalar, \( Lb \) and \( Ub \) are the corresponding lower and upper bounds of \( x \). \( C(x) \leq 0 \) and \( C_{eq}(x) = 0 \) are the possible inequality and equality constraints, respectively.

The second step is to determine the equivalent test plan based on Definitions 1 or 2 using formulations (2) or (3), respectively. Formulation (2) is given as follows,

\[
\begin{align*}
\text{Min} & \quad \Pi_j(y) \\
\text{s.t.} & \quad |f_B(x) - f_E(y)| \leq \delta, \quad \Pi_j(x) - \Pi_j(y) = 0, \\
& \quad Lb' \leq y \leq Ub', \quad C'(y) \leq 0, \quad C_{eq}'(y) = 0,
\end{align*}
\]

where \( f_B(x) \) and \( f_E(y) \) are the base and equivalent objective functions on reliability prediction, respectively and \( y \) is the decision variable of the equivalent test plan, \( \Pi() \) represents the constraint of the total number of test units, expected number of failures or the test time. If \( \Pi_j(y) \) is the total number of test units, \( \Pi_j(y) \) can be the censoring time under Type-I censoring or expected number of failures under Type-II censoring and vice versa. The idea is to set the allowed difference between objective values as a constraint as well as seek other merits.

Similarly, based on Definition 2, the optimal equivalent test plan can be determined as,

\[
\begin{align*}
\text{Min} & \quad \Pi_j(y) \\
\text{s.t.} & \quad f_B(x) - f_E(y) = 0, \quad |\Pi_j(x) - \Pi_j(y)| \leq \delta, \\
& \quad Lb' \leq y \leq Ub', \quad C'(y) \leq 0, \quad C_{eq}'(y) = 0.
\end{align*}
\]

We demonstrate these methods based on equivalent step-stress and ramp-stress test plans and the baseline constant-stress test plan.

4. Assumptions and the Failure Time Model

We assume the lifetimes of the test units are statistically independent and follow exponential distribution with a hazard rate function \( h_0(t) = \lambda, \lambda > 0 \). The applied stress affects the lifetime of a test unit through Proportional Hazard (PH) model. According to the PH assumption, the hazard function of the test units under test stress \( z \) is given by
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\[ h(t; z) = \lambda \exp(\beta z), \]

where \( \beta \) is the coefficient that reflects the effect of the stress.

Therefore the reliability function under stress \( z \) is expressed as

\[ R(t; z) = e^{-H(t; z)} = \exp[-\lambda t \exp(\beta z)], \quad (4) \]

and the corresponding failure time distribution function is

\[ f(t; z) = \lambda \exp[\beta z - \lambda t \exp(\beta z)]. \quad (5) \]

In order to investigate the life-stress relationship under time-varying stresses (piece-wise continuous over time), we extend the PH model based on the CE assumption: 1) the remaining life of a test unit depends only on current cumulative fraction of damage and the current stress regardless how the fraction is accumulated; 2) If held at the current stress, survivors fail according to the cumulative distribution for that stress, but starting at the previously cumulative damage. This assumption results in a joint cumulative failure function by horizontally shifting the individual cumulative failure function at the time that stress level changes. This can be explained by Figure 2 where \( \Psi(t; z_s) \) denotes the PDF of failure time for units tested at constant-stress \( z_s = L, H, (L \text{ and } H \text{ correspond to low and high stress respectively}), \) \( t_1 \) is the time that stress level changes, and \( t_1 - \epsilon \) is the time shift. Note that \( \epsilon \) can be solved using the equality \( \Psi(t; z_H) = \Psi(t_1; z_L) \). Since the cumulative failure function is one to one correspondence to the cumulative hazard function, the CE assumption is also directly applicable to the cumulative hazard function.

Suppose a time-varying stress \( z(t) \) is piece-wise continuous, then it can be approximated by a multi-step stress, e.g. a ramp-stress can be approximated by a step-stress as shown in Figure 3. We apply the CE and PH assumptions at every time increment on the cumulative hazard function and obtain a corresponding time shift. Then the cumulative time shift for time period \( t \) is given by

\[ t^* = \int_0^t \tau \frac{H_0(\tau + t^* (\tau))\beta}{\lambda_0(\tau + t^* (\tau))} d\tau, \]

and the cumulative hazard function is given by

\[ H(t; z) = H_0(t + t^*) \exp(\beta z(t)). \quad (6) \]
If there are $i$ jumps associated with $z(t)$ at $\tau_i$ respectively then

$$H(t; z) = H_0(t + t^* + \sum_{i} \delta_i) \exp(\beta z(t)),$$

(7)

where $\delta_i = H_0^{-1}(H_0(\tau_i + t^*) \exp(-\beta \Delta \tau_i) - t^* - \tau_i)$.

Detailed derivations of Equations (6) and (7) are given in Appendices A and C respectively.

![Figure 3. Approximation of a ramp-stress by a step-stress.](image)

5. Formulation of Baseline and Equivalent ALT Plans

The optimum baseline constant-stress ALT plan is designed under Type-I censoring with a predetermined censoring time $\tau$. Three stress levels are used as shown in Figure 4. The high stress level is chosen to be the highest value $z_H$ (normalized value). The medium level $z_M = (z_L + z_H) / 2$ is the midway between the low level $z_L$ and the high level $z_H$. The value of the low stress level is a decision variable. The allocation of test units to the low, medium and high stress levels follows the 4:2:1 ratio. This unequal allocation is a compromise that extrapolates reasonably well and results in optimum design of test plans under constant-stress loading [10]. The optimal test plan in terms of the low stress level $0 < z_L < 1$ is obtained such that the MLE of time to $1\%$ ($q = 0.01$ quantile) failure at the normal operating condition $z_P = 0$ is minimized. The total number of available test units is $N_B$. The expected number of failures at the low stress level is required to be greater than or equal to $N_B p_L$, where $p_L$ is a fraction of the test units allocated to the low stress level.

![Figure 4. The baseline constant-stress.](image)
Under Type-I censoring and the assumed failure time model, the log likelihood function of an observation at stress level \( z_k(k = L, M, H) \) is

\[
L(\lambda, \beta; z_k) = I[\ln(\lambda) + \beta z_k - \lambda t \exp(\beta z_k)] - (1 - I)\lambda \tau \exp(\beta z_k)
\]

where \( I \) is an indicator function defined by

\[
I = \begin{cases} 
1, & \text{if } t \leq \tau \text{ (failure)}. \\
0, & \text{otherwise}.
\end{cases}
\]

By taking the second derivative of the log likelihood function with respect to the unknown parameters and taking the negative expectation, we can obtain the elements of the Fisher information matrix. Let \( F_k \) be the Fisher information matrix of observations corresponding to stress level \( z_k(k = L, M, H) \) which is given by

\[
F_k = N_0 p_k \begin{bmatrix}
E \left[ \frac{\partial^2 L}{\partial \lambda^2} \right] & E \left[ -\frac{\partial^2 L}{\partial \lambda \partial \beta} \right] \\
E \left[ -\frac{\partial^2 L}{\partial \lambda \partial \beta} \right] & E \left[ \frac{\partial^2 L}{\partial \beta^2} \right]
\end{bmatrix},
\]

where

\[
E \left[ -\frac{\partial^2 L}{\partial \lambda^2} \right] = \frac{1}{\lambda^2} \{1 - \exp[-\lambda \tau \exp(\beta z_k)]\},
\]

\[
E \left[ -\frac{\partial^2 L}{\partial \beta^2} \right] = z^2 \{1 - \exp[-\lambda \tau \exp(\beta z_k)]\},
\]

\[
E \left[ -\frac{\partial^2 L}{\partial \lambda \partial \beta} \right] = \frac{z}{\lambda} \{1 - \exp[-\lambda \tau \exp(\beta z_k)]\}.
\]

The total information matrix is given by \( F_B = \sum_{k=L,M,H} F_k \).

Let \( t_q(z_D) \) be the time to the \( q \)-th quantile failure at normal operating conditions \( z_D \), then from Equation (4) we solve

\[
t_q(z_D) = \frac{\ln(1 - q)}{-\lambda \exp(\beta z_D)}.
\]

The asymptotic variance of the MLE \( \hat{t}_q(\hat{\lambda}, \hat{\beta}; z_D) \) at normal operating conditions \( z_D \) is given by

\[
\text{Asvar}[\hat{t}_q(\hat{\lambda}, \hat{\beta}; z_D)] = \begin{bmatrix}
\frac{\partial \hat{t}_q(\hat{\lambda}, \hat{\beta}; z_D)}{\partial \lambda} \\
\frac{\partial \hat{t}_q(\hat{\lambda}, \hat{\beta}; z_D)}{\partial \beta}
\end{bmatrix}^T \cdot F_B \cdot \begin{bmatrix}
\frac{\partial \hat{t}_q(\hat{\lambda}, \hat{\beta}; z_D)}{\partial \lambda} \\
\frac{\partial \hat{t}_q(\hat{\lambda}, \hat{\beta}; z_D)}{\partial \beta}
\end{bmatrix},
\]

where
\[
\frac{\partial \hat{t}_q(\hat{\lambda}, \hat{\beta}; z_D)}{\partial \lambda} = \frac{\ln(1-q)}{\lambda^2 \exp(\beta z_D)},
\]
(9)
\[
\frac{\partial \hat{t}_q(\hat{\lambda}, \hat{\beta}; z_D)}{\partial \beta} = z_D \ln(1-q) \frac{1}{\lambda \exp(\beta z_D)},
\]
(10)
and \( F_B^{-1} \) is the inverse of the Fisher information matrix.

Optimal baseline test plan is obtained by solving the following optimization problem
\[
\text{Min } f_B(x) = \text{Asvar}[\hat{t}_q(\hat{\lambda}, \hat{\beta}; z_D)]
\]
(11)
s.t. \( 0 < z_L < 1, \ p_H = \frac{1}{7}, \ p_M = \frac{2}{7}, \ p_L = \frac{4}{7}, \ N_B p_L [1 - R(\tau; z_L)] \geq \pi N_B p_L, \)
where the decision variable is the low stress level \( x = z_L \), the constraint \( N_B p_L [1 - R(\tau; z_L)] \geq \pi N_B p_L \) ensures the minimum number of failures at the low stress level.

Step-stress (shown in Figure 5) is often used in life testing in order to shorten the test duration. According to Definition 1 and the approach for determining optimal equivalent test plan, we present two formulations (two different objectives) for determining the optimal equivalent test plan under step-stress.

Figure 5. A simple step-stress.

**Formulation 1:** The objective is to minimize the censoring time \( \tau_2 \) under the step-stress test using the same number of test units as that of the baseline test plan.
\[
\text{Min } \tau_2(y)
\]
(12)
s.t. \( |f_B(x) - f_r(y)| \leq \delta, \ N_s - N_B = 0, \ N_s \Psi(\tau_1; z_L) \geq \pi N_s, \)
\[ 0 < z_L < 1, \ N_s \Psi(\tau_1; z_L) \geq \pi N_s, \tau_1 < \tau_2, \]
where \( f_r(y) = \text{Asvar}[\hat{t}_q(\hat{\lambda}, \hat{\beta}; z_D)] \), the decision variables are the low stress level \( z_L \) and time to change the stress level \( \tau_1 \) represented by
\[
y = \begin{bmatrix} z_L \\ \tau_1 \end{bmatrix}.
\]
In formulation (12), the constraint \(|f_B(x) - f_s(y)| \leq \delta\) maintains that the absolute difference between the values of the objective functions is less than or equal to \(\delta (\delta \geq 0)\). The constraint \(N_s - N_B = 0\) ensures that the total number of test units under step-stress equals that of the baseline test. Likewise, the constraint \(N_s \Psi(\tau_1; z_L) \geq \pi N_s\) ensures minimum expected number of failures at the low stress level under step-stress is greater than or equal to a fraction of the total test units. Such an optimal equivalent test plan intends to reduce the test time as well as obtain equivalent accuracy of reliability prediction as that of the constant-stress test plan.

**Formulation 2:** The objective is to minimize the total number of test units under step-stress test using the same censoring time as that of the baseline test.

\[
\text{Min } N_s(y) \quad (13)
\]

s.t. \(f_B(x) - f_s(y) \leq \delta, \quad \tau_2 - \tau = 0, \quad \tau_1 < \tau_2, \quad 0 < z_L < 1, \quad z_H = 1, \quad N_s \Psi(\tau_1; z_L) \geq \pi N_s,\)

where \(\tau\) is the censoring time of the baseline test plan.

According to Definition 2 we propose two formulations for determining optimal equivalent step-stress test plan as follows.

**Formulation 1:**

\[
\text{Min } \tau_2(y) \quad (14)
\]

s.t. \(f_B(x) - f_s(y) = 0, \quad |\tau_1 - \tau| \leq \delta, \quad 0 < z_L < 1, \quad z_H = 1, \quad \tau_1 < \tau_2, \quad N_s \Psi(\tau_1; z_L) \geq \pi N_s.\)

**Formulation 2:**

\[
\text{Min } N_s(y) \quad (15)
\]

s.t. \(f_B(x) - f_s(y) = 0, \quad |\tau_1 - \tau| \leq \delta, \quad 0 < z_L < 1, \quad z_H = 1, \quad \tau_1 < \tau_2, \quad N_s \Psi(\tau_1; z_L) \geq \pi N_s.\)

The Fisher information matrix to calculate \(f_s(y) = \text{Asvar}[^\hat{\lambda}, \hat{\beta}; z_D]\) in equations (12)-(15) is given in Appendix B.

The ramp-stress shown in Figure 6 is a stress loading that can further reduce the test time than step-stress. We now explore the design of equivalent ramp-stress test plans.

![Figure 6. The ramp-stress.](image)
According to Definition 1, we propose two formulations for determining equivalent ramp-stress test plans.

**Formulation 1:** The objective is to minimize the censoring time \( \tau_R \) under equivalent ramp-stress using the same number of test units as that of the baseline test.

\[
\text{Min } \tau_R(y) \\
\text{s.t. } |f_B(x) - f_R(y)| \leq \delta, \quad N_R - N = 0, \quad k \leq k_U, \quad k\tau_R \leq 1, \quad N_R \Psi(\tau_R; z(\tau_R)) \geq \pi N_R,
\]

where \( f_R(y) = \text{Asvar}[\hat{\tau}_q(\hat{\lambda}, \hat{\beta}; z)] \), the decision variable is the normalized ramp-rate \( y = k \).

It should be noted that in the following formulations for equivalent ramp-stress test plan. The constraint \( |f_B(x) - f_R(y)| \leq \delta \) ensures that the absolute difference between the values of the objective functions is within \( \delta (\delta \geq 0) \). The constraint \( N_R - N_B = 0 \) ensures that the total number of test units equals that of the baseline test. The constraint \( k\tau_R \leq 1 \) ensures that the highest test stress is not greater than or equal to the maximum allowed stress level \( z_H \) whose normalized value is 1. \( N_R \Psi(\tau_R; z(\tau_R)) \geq \pi N_R \) ensures minimum expected number of failures under ramp-stress is greater than or equal to \( \pi \) fraction of the total test units. In addition, we specify an upper bound for the ramp rate \( k \leq k_U \) in order to avoid different failure modes other than those occur at design stress.

Formulation 2 is similar to Formulation 1, but the objective is the minimization of the number of total test units \( N_R(y) \) and the constraint of \( N_R - N = 0 \) is replaced by \( \tau_R - \tau = 0 \).

Similarly, based on Definition 2 we also propose two formulations for determining optimal equivalent ramp-stress test plans.

**Formulation 1:**

\[
\text{Min } \tau_R(y) \\
\text{s.t. } f_B(x) - f_R(y) = 0, \quad |N_R - N_B| \leq \delta, \quad k \leq k_U, \quad k\tau_R \leq 1, \quad N_R \Psi(\tau_R; z(\tau_R)) \geq \pi N_R.
\]

Formulation 2 is similar to Formulation 1, but the objective is the minimum number of total test units \( N_R(y) \) and the constraint of \( |N_R - N_B| \leq \delta \) is replaced by \( |\tau_R - \tau| \leq \delta \).

6. **Numerical Examples**

In this section we present numerical examples for determining equivalent test plans based on Definition 1. Suppose the baseline accelerated life testing is to be carried out at three constant-voltage levels for MOS devices in order to estimate its time to 1% quantile failure at normal operating voltage conditions \( z_D = 2V \). The test needs to be completed in 300 hours. The total number of units available for testing is 200. To avoid inducing failure modes different from those expected at the design stress level, it has been determined through engineering judgment, that the highest voltage level should not exceed \( z_H = 5V \). The stress level is normalized to the range of \([0, 1]\). The required minimum number of failures for the low stress level is 30% of test units allocated to that level.

Some experiments are conducted to obtain a set of initial values of the unknown model parameters. These values are then normalized as \( \hat{\lambda} = 0.0015, \quad \hat{\beta} = 6.2 \). The decision variable is the low stress level \( z_L \). The optimal value of the decision variable is determined by solving
the nonlinear optimization problem with linear and nonlinear constraints, formulation (11). We use Matlab nonlinear constraint solver, \textit{fmincon}, to solve this optimization problem. The optimum normalized low stress level is 0.1139 which is equivalent to $z_L = 2.3417V$. The corresponding asymptotic variance of time to 1% failure at design stress is 0.8082.

Under step-voltage and ramp-voltage test, the first objective for determining equivalent test plans is to minimize the test duration while achieving equivalent asymptotic variance values of time to 1% failure to that of the baseline test. The efficiency of the equivalent plan is measured by the percentage of reduction in the test time.

To obtain the equivalent test plans for minimum censoring time we follow and evaluate the formulations (12) and (16) for equivalent step-voltage test plan and ramp-voltage test plan, respectively. The allowed absolute difference between the asymptotic variances is less or equal to 0.01, i.e. $\delta = 0.01 = 1\%$. We use the same number of test units of the baseline test plan ($N_S = N_R = N_B = 200$). We set the upper bound of the ramp-rate as 0.01V/hr. The obtained equivalent test plans are given in Table 1.

<table>
<thead>
<tr>
<th>Test plan parameters</th>
<th>Baseline const.-voltage</th>
<th>Equivalent step-voltage</th>
<th>Equivalent ramp-voltage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Obj. values</td>
<td>0.8082</td>
<td>0.8012 ($\delta = 0.87%$)</td>
<td>0.8044 ($\delta = 0.47%$)</td>
</tr>
<tr>
<td>Test time (hrs)</td>
<td>300</td>
<td>110</td>
<td>43.2</td>
</tr>
<tr>
<td>Test time reduction</td>
<td>--</td>
<td>63.33%</td>
<td>85.6%</td>
</tr>
<tr>
<td>Total number of units</td>
<td>200</td>
<td>200</td>
<td>200</td>
</tr>
</tbody>
</table>

We observe that the step-voltage test significantly reduces the test time. The time reduction relative to the baseline test plan is 63.33% while the difference between the asymptotic variances is less than 1%. The ramp-voltage test plan reduces the test time further. The time reduction relative to the baseline test plan is 85.6% while the difference between the asymptotic variance is less than 0.5%. This shows that it is feasible to design equivalent and yet efficient ALT plans having the same accuracy of reliability prediction.

When the cost per unit is high, it is important to reduce the number of test units used in accelerated life testing. Therefore, the second objective is to minimize the total number of test units under step-voltage test and ramp-voltage test while achieving equivalent accuracy of reliability prediction to that of the baseline test. The efficiency of the equivalent plan is measured by the percentage of reduction in the number of test units.

To obtain the equivalent test plans for minimum number of test units we follow and evaluate formulation (13) for step-voltage test plan. Similarly, we follow and evaluate formulation (16) with the objective function replaced by the minimum number of total test units $N_R(y)$ and the constraint of $N_R - N = 0$ replaced by $\tau_R - \tau = 0$ for the equivalent ramp-voltage test plan. The parameters of equivalent test plan are presented in Table 2.

<table>
<thead>
<tr>
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<th>Equivalent step-voltage</th>
<th>Equivalent ramp-voltage</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
<tr>
<td>Total number of units</td>
<td>200</td>
<td>200</td>
<td>200</td>
</tr>
</tbody>
</table>

Again, we observe that the step-voltage test significantly reduces the required number
of test units. The reduction relative to the baseline test plan is 40.5% while the difference between the objective functions is less than 0.5%. The ramp-voltage test plan also reduces the required number of test units even further. The reduction relative to the baseline test plan is 72% while the difference between the objective functions is less than 1%. This confirms that we can design equivalent and yet economical ALT plans having the same accuracy of reliability prediction.

### Table 2. Equivalent test plans (minimum number of test units).

<table>
<thead>
<tr>
<th>Test plan parameters</th>
<th>Baseline const.-voltage</th>
<th>Equivalent step-voltage</th>
<th>Equivalent ramp-voltage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Obj. values</td>
<td>0.8082</td>
<td>0.8111 (δ = 0.36%)</td>
<td>0.8017 (δ = 0.86%)</td>
</tr>
<tr>
<td>Censoring time (hrs)</td>
<td>300</td>
<td>300</td>
<td>300</td>
</tr>
<tr>
<td>Total number of units</td>
<td>200</td>
<td>119</td>
<td>56</td>
</tr>
<tr>
<td>Test units reduction</td>
<td>--</td>
<td>40.5%</td>
<td>72%</td>
</tr>
</tbody>
</table>

### 7. Conclusions and Future Work

In this paper, we investigate the equivalency of ALT plans involving different stress loadings. We propose four definitions of equivalency in order to design equivalent ALT plan. Based on Definitions 1 and 2, we determine optimal equivalent ALT plans under the step-stress and the ramp-stress to the baseline constant-stress ALT plan. The objective is to shorten the test duration or reduce the number of test units without any significant errors in reliability predictions. Numerical examples demonstrate the feasibility of such equivalent ALT plans under different stress loadings. This has significant practical and economical impacts as it enables reliability practitioners to choose the appropriate ALT plan to accommodate resource and duration of the test constraints.

Equivalent test plans under a multi-stress multi-step test where the stress levels can be changed at different times using different loading sequences is also an interesting and challenging problem. ALT can be used to obtain not only failure observations but also test unit’s degradation information. In addition, definition of equivalency can be extended to accelerated degradation test. New approaches to determine optimal equivalent accelerated degradation test plans based on the definition of equivalency need be investigated.

### References

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Appendix A

Under the CE assumption, the cumulative hazard rate function at the time that the stress is changing has the following relationship:

\[ H(t + t^*; z_{-1}) = H(t + t^* + \Delta t_i; z_{-1} + \Delta z_i) = H(t + t^* + \Delta t_i; z_i) \]

Using Taylor expansion

\[ \frac{\partial H(\xi; \zeta)}{\partial t} |_{t=t^*+\Delta t_i, \xi=z_{-1}+\Delta z_i} \Delta t_i + \frac{\partial H(\xi; \zeta)}{\partial \xi} |_{t=t^*+\Delta t_i, \xi=z_{-1}+\Delta z_i} \Delta z_i = 0. \]
Under the PH assumption

\[
\frac{\Delta t_i}{\Delta z_i} = - \left( \frac{\partial H(z_i; \xi)}{\partial z} \right)_{\xi = z_i} = - \frac{H_0(t + t^*) \exp(\beta z)}{\lambda_0(t + t^*) \exp(\beta z)} = - \frac{H_0(t + t^*) \beta}{\lambda_0(t + t^*)},
\]

\[
\Delta t_i = - \frac{H_0(t + t^*) \beta}{\lambda_0(t + t^*)} \Delta z_i, \text{ and } t^* = \sum_i \Delta t_i.
\]

If stress \( z(t) \) is continuous, then we can take the limit of \( \Delta z_i \to 0 \) and obtain

\[
t^* = \int_0^t \frac{H_0(\tau + t^*(\tau)) \beta}{\lambda_0(\tau + t^*(\tau))} z(\tau) d\tau, \text{ and } H(t; z) = H_0(t + t^*) \exp(\beta z(t)).
\]

If there is a jump \( \Delta z \) associated with \( z(t) \) at time \( \tau \),

\[
H(\tau; z) = H_0(\tau + t^*(\tau)) \exp(\beta z(\tau)) = H_0(\tau + t^*_i + \delta_i) \exp(\beta(z(\tau) + \Delta z)).
\]

Then we find \( \delta_i = H_0^*(H_0(\tau_i + t^*_i) \exp(-\beta \Delta z_i) - t^*_i - \tau_i) \). Therefore, if there are \( i \) jumps associated with \( z(t) \) at time \( \tau_i \) respectively then

\[
\delta_i = H_0^*(H_0(\tau_i + t^*_i) \exp(-\beta \Delta z_i) - t^*_i - \tau_i).
\]

Appendix B

According to the proposed model Equation (7), the cumulative failure function is given by

\[
\Psi(t, z(t)) = \begin{cases} 1 - \exp[-\lambda t \exp(\beta z)], & t < \tau_i, \\ 1 - \exp[-\lambda t (\tau_i + \varepsilon) \exp(\beta z)], & t \geq \tau_i. \end{cases}
\]

The log likelihood function is

\[
L(\lambda, \beta; z_L, z_H) = I_1 I_2 \ln(\lambda) + \beta z_L - \lambda t \exp(\beta z_L)
\]

\[+ (1 - I_1) I_2 \ln(\lambda) + \beta z_H - \lambda(t - \tau_i + \varepsilon) \exp(\beta z_H)\]

\[-(1 - I_2) \lambda(\tau_i - \tau_i + \varepsilon) \exp(\beta z_H),
\]

where \( I_1 \) and \( I_2 \) are indicator functions given by

\[
I_1 = \begin{cases} 1, & \text{if } t \leq \tau_i, \text{ failure observed before time } \tau_i, \\ 0, & \text{if } t > \tau_i, \text{ otherwise.} \end{cases}
\]

\[
I_2 = \begin{cases} 1, & \text{if } t \leq \tau_2, \text{ failure observed before time } \tau_2, \\ 0, & \text{if } t > \tau_2, \text{ otherwise.} \end{cases}
\]

The Fisher information matrix is given by
The Fisher information matrix is given by

$$F_R = \sum_{k=1}^{N} F_k = \sum_{k=1}^{N} \begin{bmatrix} E \left[ -\frac{\partial^2 L}{\partial \lambda^2} \right] & E \left[ -\frac{\partial^2 L}{\partial \lambda \partial \beta} \right] \\ E \left[ -\frac{\partial^2 L}{\partial \lambda \partial \beta} \right] & E \left[ -\frac{\partial^2 L}{\partial \beta^2} \right] \end{bmatrix},$$

where

$$E \left[ -\frac{\partial^2 L}{\partial \lambda^2} \right] = \frac{1}{\lambda^2} \left\{ 1 - \exp[-\lambda (r_2 - r_1 + \varepsilon) \exp(\beta z_H)] \right\},$$

$$E \left[ -\frac{\partial^2 L}{\partial \beta^2} \right] = z_L^2 + (z_H^2 - z_L^2) \exp[-\lambda \tau_1 \exp(\beta z_L)] - x_H^2 \exp[-\lambda (r_2 - r_1 + \varepsilon) \exp(\beta z_H)],$$

$$E \left[ -\frac{\partial^2 L}{\partial \lambda \partial \beta} \right] = \frac{z_L^2}{\lambda} \left\{ 1 - \exp[-\lambda \tau_1 \exp(\beta z_L)] \right\}$$

$$+ z_H \tau_1 \exp(\beta z_L - \lambda \tau_1 \exp(\beta z_L))$$

$$+ \frac{z_H}{\lambda} \left\{ \exp[-\lambda \tau_1 \exp(\beta z_L)] - \exp[-\lambda (r_2 - r_1 + \varepsilon) \exp(\beta z_H)] \right\}.$$
\[
E \left[ -\frac{\partial L^2}{\partial \lambda \partial \beta} \right] = \int_0^\infty \left\{ \frac{1 - \exp(\beta kt)}{\beta^2 k} + \frac{t \exp(\beta kt)}{\beta} \right\} dt \\
+ \exp \left\{ \frac{\lambda}{\beta k} \left[ 1 - \exp(\beta kr) \right] \right\} \left[ \frac{1 - \exp(\beta kr)}{\beta^2 k} + \beta kr \exp(\beta kr) \right],
\]

\[
E \left[ -\frac{\partial L^2}{\partial \beta^2} \right] = \int_0^\infty \left\{ \frac{\lambda k^2 \exp(\beta kt)}{\beta} - \frac{2 \lambda t \exp(\beta kt)}{\beta^2} - \frac{2 \lambda [1 - \exp(\beta kt)]}{\beta^2 k} \right\} dt \\
+ \exp \left\{ \frac{\lambda}{\beta k} \left[ 1 - \exp(\beta kr) \right] \right\} \left\{ \frac{\lambda k^2 \exp(\beta kr)}{\beta} - \frac{2 \lambda t \exp(\beta kr)}{\beta^2} - \frac{2 \lambda [1 - \exp(\beta kr)]}{\beta^2 k} \right\}.
\]

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