A Study on Joint Availability for $k$ out of $n$ and Consecutive $k$ out of $n$ Points and Intervals

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Abstract: The performance of maintenance systems can be described by many indexes such as availability, mean up-time and mean down-time and so forth. The availability is the most important measure among them. Availability has many types, including instantaneous, steady-state and average availability, etc. In this paper, the joint availabilities for $k$ out of $n$ intervals and consecutive $k$ out of $n$ intervals under the Markov assumption are presented in recursive equations. When $n=k$, the $k$ out of $n$ and consecutive $k$ out of $n$ systems reduce to a series system, and in this situation, the corresponding results have been given by Csenki [9] and Cui et al. [17] using different methods. In contrast, the major contributions of the paper are the extensions of the known results for series systems; that is, the situations when $k<n$ have not been covered in the previous studies and will be presented in the present paper. Some numerical examples and discussions are given to illustrate the results obtained in the paper. Finally, the conclusions are summarized.

Keywords: Aggregated stochastic processes, consecutive $k$ out of $n$, joint interval availability, $k$ out of $n$, Markov repairable system.

1. Introduction

In the reliability field, although repairable systems have been widely investigated during the past several decades, the study on these systems still remains a hot topic in both theory and practice; for example, see recent publications such as Bao and Cui [1, 2], Cui et al. [15], Hawkes et al. [18] and Wang et al. [21]. In order to analyze the performance of repairable systems, many related indexes have been developed in the reliability literature; see, for example, Cui et al. [11] and the references therein. The most popular index among repairable system performance measures is the availability, which is the probability of a repairable system being in the working states or operational states in some concerned time or during specified duration. Many papers have been published on the study of availability for various repairable systems; see, for example, Cui and Xie [10, 12]. Every formula for availability is given using specific techniques under specified maintenance models, so that different maintenance models yield different availability formulas even for the same repairable system. For maintenance models, the readers may be referred to Cui [14] for a literature summary. On the other hand, the different techniques may result in the different degree of difficulty for derivations of the availability formulas. There are many types of availability such as instantaneous or steady-state availability, and so on, and each type availability depicts some parts of performances of repairable systems. Recently, the aggregated stochastic processes have been used to analyzing repairable systems in which
the Ion channel modeling techniques (see Colquhoun and Hawkes [5]) were used. For example, Zheng et al. [22] discussed a single-unit Markov repairable system with repair time omission, and Cui et al. [13] considered the several indexes including availability for aggregated Markov repairable system with history-dependent up and down states. Zheng et al. [23] and Wang and Cui [20] extended the history-dependent case into Semi-Markov repairable systems, Liu et al. [19] studied the interval reliability for aggregated Markov repairable system with repair time omission, and Cui et al. [16] studied the reliability for two-part partition of states for aggregated Markov repairable systems, and among others.

In the present paper, we shall consider the joint interval availability which is an extension of interval availability. The interval availability is the probability that at a specified time, the system is operating and will continue to operate for an interval of duration, which is slightly different from the definition of interval reliability given in Barlow and Proschan [3]. The Ion-channel modeling techniques will be used in the paper to study the joint availability for $k$ out of $n$ and consecutive $k$ out of $n$ intervals and points. Csenki [6, 8] discussed the interval reliability under the Semi-Markov assumption in which the interval reliability satisfies a set of integral equations which can be solved numerically. Csenki [7] first considered the joint availability under the semi-Markov assumptions and proved that the joint availability also satisfies an integral equation which can be solved numerically. Csenki [9] gave a closed form expression under the Markov assumption for the joint interval reliability and established it by the induction. Cui et al. [17] named this joint interval reliability as joint availability and presented different closed form expressions under the same assumption for the joint interval availability. Cui et al. [17] also gave the closed form expressions for the mixed multi-point-interval availability, and derived some relationship and inequalities among these point and interval availabilities. In this paper, the joint availabilities for $k$ out of $n$ intervals and consecutive $k$ out of $n$ intervals under the Markov assumption are presented in recursive equations. When $n = k$, the $k$ out of $n$ and consecutive $k$ out of $n$ systems reduce to a series system, and in this situation, the corresponding results have been given by Csenki [9] and Cui et al. [17] using different methods. A major contribution of the present paper is the extensions of the known results for series systems to the situations when $k < n$ that have not yet been studied in the previous studies. The current work may enrich the contents along this direction. The practical applications for this problem may be found in Csenki [9] and in the introduction section of Cui et al. [17]. We here give an illustrative practical application example. Consider a computer center consisting of many servers, and people may be interested in working probabilities of servers that can be up continuously for certain periods of time. For example, if people want to know how much the probability that the servers are in working states continuously in 6 days in a week would be, then the results that will be obtained in the paper may be useful. In fact, many practical problems in reliability and risk can be modeled as joint availability problems.

The rest of the paper is organized as follows. Section 2 gives assumptions on the model and some related basic results of Markov repairable systems to be used in the paper. The main results for $k$ out of $n$ and consecutive $k$ out of $n$ intervals and points are presented in Sections 3 and 4, respectively. In Section 5, some numerical examples and related discussions are given to illustrate the results obtained in the paper. Finally, the conclusions are summarized in Section 6.

2. Models and Related Basic Preliminaries

Let a repairable system $\{X(t), t \geq 0\}$ be a continuous time, irreducible and
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homogeneous Markov process with finite state space $S$ which can be partitioned into two sets: working states $W$ and failure states $F$, i.e., $S = W \cup F$. The corresponding transition rate matrix $Q$ can be divided into four parts in terms of the partition of state space, i.e.,

$$Q = \begin{pmatrix} Q_{WW} & Q_{WF} \\ Q_{FW} & Q_{FF} \end{pmatrix},$$

and we can define a matrix as

$$P_{ww}(t) \triangleq \{ P\{X(t) = j, X(u) \in W, u \leq t | X(0) = i\}, i, j \in W, $$

and then, in terms of the results given in Colquhoun and Hawkes [5], we have

$$P_{ww}(t) = \exp(Q_{ww}t),$$

where $P_{ww}(t)$ denotes the probability that the repairable system remains within the working states $W$ throughout the time from 0 to $t$. This quantity can be also expressed in terms of an exponential matrix as the usual way, i.e.,

$$\exp(Q_{ww}t) = I + Q_{ww}t + \frac{(Q_{ww}t)^2}{2!} + \frac{(Q_{ww}t)^3}{3!} + \cdots.$$

The questions we shall consider are as follows. Given $n$ intervals: $[a_1, b_1], \ldots, [a_n, b_n]$ ($0 \leq a_1 < a_2 < b_2 < \cdots < a_n < b_n$), what is the probability that Markov process $\{X(t), t \geq 0\}$ is in the working states for at least $k$ intervals or consecutive $k$ intervals and stays in the working states continuously during each of these $k$ intervals? We call this probability as a joint interval availability for $k$ out of $n$ intervals $[a_1, b_1], \ldots, [a_n, b_n]$ or consecutive $k$ out of $n$ intervals $[a_1, b_1], \ldots, [a_n, b_n]$. When all the intervals degenerate into points, i.e., $b_i = a_i (i = 1, 2, \ldots, n)$, the problem becomes a joint interval availability for $k$ out of $n$ points $a_1, a_2, \ldots, a_n$ or consecutive $k$ out of $n$ points $a_1, a_2, \ldots, a_n$. When only some of the intervals degenerate into points, the results also include the mixed multiple points and intervals. When $n = 1$, the problem is an interval reliability problem, which was discussed by Csenki [6]. When $k = n$, the problem was considered by Csenki [9] and Cui et al. [17] via different methods. When $k < n$, the problem becomes new one that will be discussed in the present paper. The contributions of our paper are to fill in this gap and to unify all results into one formula.

In practical maintenance situations, people are often interested in the following problem: what is the probability that a repairable system is in working states for at least $k$ or consecutive $k$ intervals of $[a_1, b_1], \ldots, [a_n, b_n]$? Here the repairable system being in working states in one interval $[a_i, b_i]$ means that the repairable system stays in the working states $W$ throughout time interval $[a_i, b_i]$. The problems presented here are more common in the maintenance field as well in other areas. If the underlying process can be depicted by the Markov process, then the results obtained in the paper may be useful in analyzing the probability that the process or procedure stays in some kind of states for several intervals or consecutive intervals.

3. Main Results for $k$ Out of $n$ Intervals

In this section, $k$ out of $n$ intervals is considered as follows $(0 \leq k \leq n)$.

Let $R(k,[a_1, b_1], \ldots, [a_n, b_n]; \pi_0)(R(k,t_1, \ldots, t_n; \pi_0))$ be the availability that the Markov repairable system $\{X(t), t \geq 0\}$ is in working states for at least $k$ out of $n$ intervals
Given a Markov repairable system denoted as \( \{X(t), t \geq 0\} \) with finite state \( S = W \cup F \) (\( W \cap F = \phi \)) and transition rate matrix \( Q \), the initial probability is \( \pi_0 \), and we have

\[
R(k, [a_1, b_1], \ldots, [a_n, b_n]; \pi_0) = \pi_0 P(X(t) \in W) \text{ for at least } k \text{ intervals of } [a_1, b_1], \ldots, [a_n, b_n]; \pi_0, \]

We say that the stochastic process is available in an interval \([a_i, b_i] \), if \( X(t) \in W \) for all \( t \in [a_i, b_i] \), and this can be measured by the probability known as availability.

**Theorem 1.** Given a Markov repairable system denoted as \( \{X(t), t \geq 0\} \) with finite state \( S = W \cup F \) (\( W \cap F = \phi \)) and transition rate matrix \( Q \), the initial probability is \( \pi_0 \), and we have

\[
R(k, [a_1, b_1], \ldots, [a_n, b_n]; \pi_0) = R(k-1, [a_1, b_1, b_2-b_1], \ldots, [a_n-b_n-b_1]; \pi_0) + R(k, [a_1, b_1, b_2-b_1], \ldots, [a_n-b_n]; \pi_0),
\]

\[
R(0, [a_1, b_1], \ldots, [a_n, b_n]; \pi_0) = \pi_0 e^{Q t} I_S, \quad \text{(for any positive integer } m),
\]

with the boundary conditions

\[
R(j, [a_1, b_1], \ldots, [\alpha_j, \beta_j]; \pi) = \prod_{i=1}^{j-1} \left( e^{Q (a_i, b_i) - \Delta_i} e^{Q (\alpha_i, \beta_i)} E^{T} \right) e^{Q (\alpha_j, \beta_j)} E^{T} \pi, \quad j = 1, 2, \ldots, k,
\]

and \( R(0, [\alpha, \beta]; \pi) = \pi_0 e^{Q t} I_S, \) for any intervals \([\alpha_1, \beta_1], \ldots, [\alpha_j, \beta_j]\) and \([\alpha, \beta]\),

\[
\pi_i^1 = \pi_0 e^{Q t} E^{T} \pi, \quad E = \begin{pmatrix} I_{|W|} & |W| \\ 0 & |F| \end{pmatrix}, \quad \pi_0^1 = \pi_0 e^{Q t} - \pi_i^1,
\]

where \( \Delta_i = b_i - a_i \) (\( i = 1, 2, \ldots, n \) ), \( b_0 = 0 \), and \( T \) denotes the operation of transpose, \( I_\Omega \), \( \Omega \in \{S, W, F\} \), is a column vector with all ones in dimension \( |\Omega| \).

**Proof.**

\[
R(k, [a_1, b_1], \ldots, [a_n, b_n]; \pi_0) = R(k-1, [a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]; \pi_0, X(t) \text{ is available in } [a_1, b_1]) + R(k, [a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]; \pi_0, X(t) \text{ is not available in } [a_1, b_1])
\]

\[
= R(k-1, [a_1, b_1, b_2-b_1], \ldots, [a_n-b_n-b_1]; \pi_0) + R(k-1, [a_2-b_2, b_2-b_1], \ldots, [a_n-b_n-b_1]; \pi_0).
\]

The equation for \( R(j, [\alpha_1, \beta_1], \ldots, [\alpha_j, \beta_j]; \pi) \) follows from the work of Cui et al. [17]. The equation for \( R(0, [\alpha, \beta]; \pi) \) is obvious, because \( \pi_0 e^{Q t} I_S \) is the probability that the system evolves to point \( \beta \) regardless of whether the system is available or not in the interval \([\alpha, \beta]\). The last equation of the proof of Theorem follows from the following facts: the first term is due to the fact that \( X(t) \) is available in \([a_1, b_1]\), and each interval must be subtracted by the first interval \([a_1, b_1]\) and the initial probability is changed from \( \pi_0 \) to \( \pi_0 e^{Q t} \), and the second term is resulted similarly.

The recursive equations for the initial conditions are based on the probability whether the system is available or not in the first interval \([a_1, b_1]\).

The matrix \( E \) and vector \( I_\Omega \) are needed because of multiplications among matrices.
Note that:

1) The recursive paths are \((n,k) \rightarrow (n-1,k), (n,k) \rightarrow (n-1,k-1)\)
and \((m,0) \rightarrow (m-1,0) \rightarrow \cdots \rightarrow (1,0)\) where the point \((x,y)\) denotes the \(y\) out of \(x\) intervals.

2) The complexity of computations is given as follows. First we consider the number of operations in decompositions. The decomposition can be classified into two cases: (i) \(n \geq 2k\), (ii) \(n < 2k\). Both decompositions can be done in the areas shown in Figure 1.

![Figure 1. Decomposition areas.](image)

The number of operations in decomposition for case (i), i.e. when \(n \geq 2k\), we have
\[
\begin{align*}
2 + 2^2 + \cdots + 2^k &= O(2^k), \quad \text{in area 1}, \\
2^{k+1} - 1 + 2^{k+2} + 1 + \cdots + 2^{n-k} + 2n - 4k - 3 &= O(2^{n-k}), \quad \text{in area 2}, \\
2^{n-2k+1} + \cdots + 2^{n-2k+k-1} &= O(2^{n-k}), \quad \text{in area 3}.
\end{align*}
\]

To sum up the three numbers, we have that the complexity of operations in decomposition is \(O(2^{n-k})\) when \(n \geq 2k\). Similarly, we have, for case (ii), i.e., \(n < 2k\),
\[
\begin{align*}
2 + 2^2 + \cdots + 2^{n-k} &= O(2^{n-k}), \quad \text{in area 1}, \\
2^{n-k} + 2^{n-k+1} + \cdots + 2^{2k-n} &= O(2^{2k-n}), \quad \text{in area 2}, \\
2^{2k-n+1} + \cdots + 2^k &= O(2^{n-k}), \quad \text{in area 3}.
\end{align*}
\]

Then the complexity of operations in decomposition is \(O(2^{\max(n,k,2k-n)})\) when \(n < 2k\). After the decompositions, we only need to compute the boundary values \(R(0,[\alpha,\beta]; \pi)\) and \(R(j,[\alpha_j,\beta_j],\ldots,[\alpha,\beta]; \pi)\) \((j = 1, 2, \ldots, k)\). In each decomposition, we know that the maximum primary operations (addition, subtraction and multiplication) are \(O(n)\) if we treat a matrix operation as a primary one. Thus the complexity of total computations for the recursive formulas is \(O(n2^{\max(n-k,2k-n)})\).

In fact, in the recursive equation presented in Theorem 1, the recursive equations for initial probability are included, which may be more important.
(3) When all intervals degenerate into points, then we have Corollary 1. In fact, when some parts of intervals degenerate into points, then we can have the availability for \( k \) out of \( n \) mixed points and intervals.

**Corollary 1.** Given a Markov repairable system denoted as \( \{X(t), t \geq 0\} \) with finite state \( S = W \cup F \ (W \cap F = \phi) \) and transition rate matrix \( Q \), the initial probability is \( \pi_0 \), and we have

\[
R(k, t_1, \ldots, t_n; \pi_0) = R(k-1, t_2 - t_1, \ldots, t_n - t_1; \pi_1^0) + R(k, t_2 - t_1, \ldots, t_n - t_1; \pi_1^0),
\]

and other related equations hold after changing the intervals into points.

4. Main Results for Consecutive \( k \) Out of \( n \) Intervals

Consecutive-\( k \) systems have been extensively studied since 1980, and there are a lot of related references; for example, see Chang et al. [4]. A linear consecutive \( k \) out of \( n \) : G system is defined as a system that consists of \( n \) components ordered in a line where the system works if and only if there exist at least consecutive \( k \) working components in the system. In the following, we shall consider a consecutive \( k \) out of \( n \) intervals \([a_1, b_1], \ldots, [a_n, b_n]\) system.

Let \( R_C(k, [a_1, b_1], \ldots, [a_n, b_n]; \pi_0) \) be the probability that the Markov repairable system \( \{X(t), t \geq 0\} \) is working states for at least consecutive \( k \) out of \( n \) intervals \([a_1, b_1], \ldots, [a_n, b_n]\) with initial probability \( \pi_0 \). Let \( L(i,n) \) be an event of the system is in the working states for at least consecutive \( k \) out of \( n \) intervals \([a_i, b_i], \ldots, [a_n, b_n]\), \( i = 1, 2, \ldots, n \).

**Lemma.** \( L(1,n) = L(2,n) \cup L(k+2,n) \bar{S}_{k+1} S_k \cdots S_1 \), where \( S_i \) be an event of the system works in interval \([a_i, b_i]\), \( i = 1, 2, \ldots, n \).

**Proof.** If \( L(2,n) \) occurs, then of course \( L(1,n) \) occurs. The only case that \( L(1,n) \) occurs but \( L(2,n) \) does not is when the system works in all first \( k \) intervals and fails in the \((k+1)\)th interval.

**Theorem 2.** Given a Markov repairable system denoted as \( \{X(t), t \geq 0\} \) with finite state \( S = W \cup F \ (W \cap F = \phi) \) and transition rate matrix \( Q \), the initial probability is \( \pi_0 \), and we have

\[
R_C(k, [a_1, b_1], \ldots, [a_n, b_n]; \pi_0) = R_C(k, [a_2, b_2], \ldots, [a_n, b_n]; \pi_0) + \left[ P\{\bar{S}_{k+1} S_k \cdots S_1\} - R_C(k, [a_{k+2}, b_{k+2}], \ldots, [a_n, b_n]; \pi_2) \right],
\]

with the boundary conditions

\[
R_C(k, [a_{n-k+1}, b_{n-k+1}], \ldots, [a_n, b_n]; \pi_0) = \pi_0 \prod_{i=n-k+1}^{n-1} e^{Q(a_i-b_i)} E e^{Q_{ww} \Lambda_e} E^T e^{Q(a_{i+1}-b_i)} E e^{Q_{ww} \Lambda_e} E^T,
\]

and

\[
R_C(k, [a_m, b_m], \ldots, [a_n, b_n]; \pi) = 0
\]

when

\[
m > n - k + 1,
\]

where

\[
\pi_2 = \pi_0 \prod_{i=1}^{k-1} \left[ e^{Q(a_i-b_i)} E e^{Q_{ww} \Lambda_e} E^T \right] e^{Q(a_{i+1}-b_i)} E e^{Q_{ww} \Lambda_e} E^T, \quad P\{\bar{S}_{k+1} S_k \cdots S_1\} = \pi_2 I_S.
\]
Proof. Because the events \( L(2,n) \) and \( L(k+2,n)S_{k+1}S_k\cdots S_1 \) are disjoint,

\[
P(L(2,n)S_{k+1}S_k\cdots S_1) = [1 - P(L(2,n)S_{k+1}S_k\cdots S_1)P(S_{k+1}S_k\cdots S_1)],
\]

and then we use Lemma above to complete the proof of Theorem 2. Here the initial probability is given by \( \pi_2 = \pi_1 e^{\mathbf{Q} \mathbf{h}_{n-1} - \mathbf{h}_{n-1} \mathbf{Q}^T} E^T \).

Note that:

1. The recursive paths are \((n,k) \rightarrow (n-1,k), (n,k) \rightarrow (n-k-1,k)\), the point \((x,y)\) denotes the consecutive \(y\) out of \(x\) intervals.

2. The complexity of computations is given as follows.

   In each decomposition, the primary operations of computations (like \( P(S_{k+1}S_k\cdots S_1) \)) are \(O(k)\) if we treat a matrix operation as a primary one, and we have the \(O(n-2k)\) terms after the decompositions. Thus the complexity of total computations for the recursive formulas is \(O(kn)\) including the computations of initial probability.

3. Similarly, when all intervals degenerate into points, then we have Corollary 2.

Corollary 2. Given a Markov repairable system denoted as \( \{X(t), t \geq 0\} \) with finite state \( S = W \cup F \) (\( W \cap F = \phi \)), transition rate matrix \( \mathbf{Q} \), and the initial probability \( \pi_0 \), we have

\[
\mathbf{R}_c(k,t_1,\ldots,t_n;\pi_0) = \mathbf{R}_c(k,t_2,\ldots,t_n;\pi_0) + [P(S_{k+1}S_k\cdots S_1) - \mathbf{R}_c(k,t_2,\ldots,t_n;\pi_2)],
\]

and other related equations hold after changing the intervals into points, where \( \mathbf{R}_c(k,t_1,\ldots,t_n;\pi_0) \) is the probability that the system works for at least consecutive \(k\) out of \(n\) points.

5. Numerical Examples and Discussions

Some numerical examples are given in this section, and are used to illustrate the results obtained in the previous sections.

We assume that the Markov process \( \{X(t), t \geq 0\} \) of the repairable system has a state space \( S = \{1,2\} = \{1\} \cup \{2\} \), i.e., the working and failure states are \( W = \{1\} \) and \( F = \{2\} \), the initial probability is \( \pi_0 = (\alpha, 1 - \alpha) \), and its transition rate matrix is

\[
\mathbf{Q} = \begin{bmatrix}
\mathbf{Q}_{WW} & \mathbf{Q}_{WF} \\
\mathbf{Q}_{FW} & \mathbf{Q}_{FF}
\end{bmatrix} = \begin{bmatrix}
-\lambda & \lambda \\
\mu & -\mu
\end{bmatrix}.
\]

Let \( \lambda = 2, \mu = 1, \alpha = 0.95 \), then we can get the following numerical results for the situations of \(k\) out of \(n\) intervals and consecutive \(k\) out of \(n\) intervals.

1. \(k\) out of \(n\) intervals \(\{a_1,b_1\},\ldots,\{a_n,b_n\}\)

   (i) \(n < 2k\)

   We take \(n=4, k=3\) for example and assume the four intervals are

   \[
   [a_1,b_1] = [0.1,0.12], [a_2,b_2] = [0.15,0.16], [a_3,b_3] = [0.2,0.25], [a_4,b_4] = [0.29,0.33].
   \]

   According to the recursive equations in Theorem 1, the recursive paths are \((4,3) \rightarrow (3,3), (4,3) \rightarrow (3,2), \cdots, (2,1) \rightarrow (1,1), (2,1) \rightarrow (1,0)\). The decomposition can be shown in Figure 2.

   Let the point \((x,y)\) denote the \(y\) out of \(x\) intervals, then the value of each point in the recursive process can be denoted as \(\mathbf{R}(x,y)\), which are listed in Table 1.
Figure 2. Decomposition for 3 out of 4 intervals.

Table 1. The value of each point in the recursive process for 3 out of 4 intervals.

<table>
<thead>
<tr>
<th>$R(x, y)$</th>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>0</td>
<td>0.5869</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.0072</td>
<td>0.5941</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.0020</td>
<td>0.5961</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>0.0076</td>
<td>0.6037</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Since the availability for $k$ out of $n$ intervals in Theorem 1 can be divided into two parts, thus the recursive process for a 3 out of 4 intervals situation can be depicted by Figure 2, in which the value of each point is decomposed into two points directed by the arrows. And the values in Table 1 have the following quantitative relations, which will further illustrate the recursive process in Theorem 1.

When $x \neq y$, the quantitative relations for elements in Table 1 are

$$R(x, y) = R(x-1, y-1) + R(x-1, y).$$

When $x = y$ or $y = 0$, $R(x, y)$ can be computed by the boundary conditions in Theorem 1. Finally, the value of $R(4,3)$ is the joint interval availability for 3 out of 4 intervals.

Taking the (1,2) entry 0.5941 in Table 1 for example, the value in row $y = 1$, column $x = 2$ equals to the value in row $y = 0$, column $x = 1$ plus the value in row $y = 1$, column $x = 1$, that is

$$R(2,1) = R(1,0) + R(1,1) = 0.5869 + 0.0072 = 0.5941.$$

Other values have the same relationship,

$$R(3,2) = R(2,1) + R(2,2) = 0.5941 + 0.0020 = 0.5961.$$  
$$R(4,3) = R(3,2) + R(3,3) = 0.5961 + 0.0076 = 0.6037.$$

Therefore, we get the joint interval availability for 3 out of 4 intervals, that is

$$R(3,[0.1,0.12],[0.15,0.16],[0.2,0.25],[0.29,0.33];\pi_0) = 0.6037.$$  

Based on the numerical results above, a comparison with Cui et al. [17] can be made. In Cui et al. [17], the availability for the same four intervals are obtained, which is 0.5009.
A Study on Joint Availability

Thus

$$R(3,[0.1,0.12],[0.15,0.16],[0.2,0.25],[0.29,0.33];\pi_0) = 0.6037 > 0.5009.$$  

In fact, the availability for four intervals equals to 4 out of 4 intervals. And it is obvious that the availability for 3 out of 4 intervals must be larger than the availability for 4 out of 4 intervals. Also the availability for five intervals can be obtained by applying Theorem 1, which is the same as the results in Cui et al. [17].

(ii) $n \geq 2k$

We take $n = 5, k = 2$ for example. We assume the four intervals are

$$[a_1, b_1] = [0.1, 0.12], [a_2, b_2] = [0.15, 0.16], [a_3, b_3] = [0.2, 0.25],$$

$$[a_4, b_4] = [0.29, 0.33], [a_5, b_5] = [0.5, 0.7]$$

According to the recursive equations in Theorem 1, the recursive paths are $(5,2) \rightarrow (4,1), (5,2) \rightarrow (4,2), \cdots, (2,1) \rightarrow (1,1), (2,1) \rightarrow (1,0)$. The decomposition can be shown in Figure 3.

In this situation, the values of each point in the recursive process for 2 out of 5 intervals can be denoted as $R_c(x, y)$, which are given in Table 2.

![Figure 3. Decomposition for 2 out of 5 intervals.](image)

<table>
<thead>
<tr>
<th>$R(x, y)$</th>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0.0124</td>
<td>0.0113</td>
<td>0.7014</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.0056</td>
<td>0.0180</td>
<td>0.0210</td>
<td>0.7132</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.0079</td>
<td>0.0163</td>
<td>0.0255</td>
<td>0.7387</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The decomposition in Figure 3 seems more complex than that in Figure 2. In Table 2, some points depend on more than one neighboring points, such as $R(2,1)$ and $R(3,1)$. Hence, some values in Table 2 can not be added by other two values directly.

In Table 2, $R(5,2) = R(4,2) + R(4,1)$, while $R(4,2)$ and $R(4,1)$ are decomposed into $R(3,1)$. Then we denote the value of $R(3,1)$ coming from $R(4,1)$ as $R^A(3,1)$, the value of $R(3,1)$ coming from $R(4,2)$ as $R^B(3,1)$, so that

$$R(4,1) = R(3,0) + R^A(3,1),$$
According to the same recursive rule, with the boundary conditions in Theorem 1, we can compute every part of each point, \( R(3,1) \) and \( R(3,2) \) can be obtained from \( R(2,0) \) and \( R(2,1) \). Then \( R(2,0) \) and \( R(2,1) \) will be decomposed into some parts, and so on. Finally the value of \( R(5,2) \) is the joint interval availability for 2 out of 5 intervals.

For instance, through computations, we get

\[
R(3,1) = 0.0118, \quad R(3,2) = 0.0092, \\
R(3,0) = 0.7014, \quad R(3,2) = 0.0163,
\]

Thus,

\[
R(4,1) = R(3,0) + R(3,1) = 0.7014 + 0.0118 = 0.7132, \\
R(4,2) = R(3,2) + R(3,1) = 0.0163 + 0.0092 = 0.0255, \\
R(3,1) = R(3,1) + R(3,1) = 0.0118 + 0.0092 = 0.0210.
\]

That is a computation process for the values in row \( y = 1 \), column \( x = 4 \), in row \( y = 1 \), column \( x = 4 \), and in row \( y = 2 \), column \( x = 4 \) in Table 2.

Finally,

\[
R(5,2) = R(4,2) + R(4,1) = 0.0255 + 0.7132 = 0.7387.
\]

Therefore, we get the joint interval availability for 2 out of 5 intervals, that is

\[
R(2,[0.1,0.12],[0.15,0.16],[0.2,0.25],[0.29,0.33],[0.5,0.7];\pi_0) = 0.7387.
\]

From the above examples, the availability for two situations of \( k \) out of \( n \) intervals can be obtained by applying Theorem 1.

The values shown in Tables 1 and 2 are detailed recursive values for \( y \) out of \( x \) in different intervals which are obtained by subtracting the first interval in each iteration, so that these values are availabilities of the repairable systems for some new intervals.

We now give another example to illustrate the situation for consecutive \( k \) out of \( n \) intervals.

(2) Consecutive \( k \) out of \( n \) intervals \([a_1, b_1], \ldots, [a_n, b_n]\)

We take \( n = 4, k = 3 \) for example. We still assume the four intervals are

\[
[0.1,0.12],[0.15,0.16],[0.2,0.25],[0.29,0.33],
\]

According to the recursive equations in Theorem 2, we can get the joint interval availability for consecutive 3 out of 4 intervals, that is

\[
R_C(3,[0.1,0.12],[0.15,0.16],[0.2,0.25],[0.29,0.33];\pi_0) = 0.5945.
\]

Therefore, it can be seen that

\[
R(3,[0.1,0.12],[0.15,0.16],[0.2,0.25],[0.29,0.33];\pi_0) = 0.6037 \\
> R_C(3,[0.1,0.12],[0.15,0.16],[0.2,0.25],[0.29,0.33];\pi_0) = 0.5945.
\]
In fact, the consecutive 3 out of 4 situation is contained in the 3 out of 4 situation, that is why the availability for 3 out of 4 intervals is larger than the availability for consecutive 3 out of 4 intervals.

6. Conclusions

In the paper, the joint interval availability has been studied for $k$ out of $n$ intervals and consecutive $k$ out of $n$ intervals under the Markov assumption, the results are presented by the recursive equations. Also the boundary conditions for the recursive equations are given.

The joint interval availability is an extension of interval availability, and the results for joint interval availability cover the special cases when the intervals degenerate into points, which also include the mixed multiple points and intervals when only some of parts of intervals degenerate into points. According to the theorems presented in the paper, the joint interval availability, the availability for $k$ out of $n$ points and consecutive $k$ out of $n$ points are obtained, due to the fact that related equations hold after changing the intervals into points.

All results obtained in the paper are the extension of the known results. The complexity of computations has been also discussed briefly for both problems. For the situation of $k$ out of $n$ intervals, two different decomposition cases are considered, which are $n \geq 2k$ and $n < 2k$, respectively. Finally, some numerical examples and discussions are given to illustrate the results obtained in the paper. Two examples for $k$ out of $n$ intervals situations are given, in which the depositions are shown by figures. In addition, the values in the recursive process are listed in tables and the quantitative relations for the values are discussed in detail.

Since the joint interval availability under the Markov assumption has been studied in the paper, the related further research may be done under the semi-Markov assumption. Besides, other indexes, including other availability measures to describe the performances of maintenance systems could be introduced and considered in the future.

References


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