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The Inverse Weibull Distribution as a Failure Model Under Various Loss Functions and Based on Progressive First-Failure Censored Data

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Abstract: In this article we consider statistical inferences about the unknown parameters of the inverse Weibull distribution based on progressively first-failure censoring using Bayesian procedures. The Bayes estimators are obtained based on both the symmetric and asymmetric (*Linex*, General Entropy and Precautionary) loss functions. There are no explicit forms for the Bayes estimators; therefore, we propose the Lindley's approximation method to compute the Bayes estimators. A comparison between these estimators and the maximum likelihood estimator (*MLE*) is provided by using extensive simulation and two criteria, namely, the bias and the mean squared error. It is concluded that the approximate Bayes estimators outperform the *MLEs* most of the time. Real life data example is provided to illustrate our proposed estimators.

Keywords: Inverse Weibull distribution, Lindley's approximation, maximum likelihood, progressive first-failure censoring.

1. Introduction

In life testing experiments, it is a common practice to cease testing before the failure of all items. This is due to the lack of funds and/or time constrains. Samples that result from such situations are called censored samples. There are several censoring methods available to experimenters, for example: type-I censoring, in which the test ceases at a pre-fixed time, and type-II censoring that allows the experiment to be terminated at a predetermined number of failures. These methods do not allow the removal of active units during the experiment. Therefore, the focus in the last few years has been on progressive censoring due to its flexibility that allows the experimenter to remove active units during the experiment. A progressively type-II censoring is a generalization of type-II censoring. Many authors have discussed inference under progressive censoring using different lifetime distributions. Among others, Cohen [12], Mann [23], Wingo [35], Balakrishnan and Sandhu [6], Aggarwala and Balakrishnan [1], Balakrishnan and Asgharzadeh [4]. For a comprehensive recent review of progressive censoring, readers may refer to Balakrishnan [2].

Johnson [18] introduced the first-failure censoring plan where the experimenter can arrange n sets, each consisting of k units, where all the $k \times n$ units can be tested simultaneously until the first failures in each n set occurs. However, in situations where the lifetime of a product was long and test facilities were limited but test material was cheap, Balasooriya [7] modified Johnson's [18] approach by testing each set one after the other

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until the first failure in each set occurred. This modified approach can save time and money.

Although this procedure indicates that a first failure censoring scheme is terminated when the first failure in each set is observed, it is desirable for researchers to be able to remove sets before the final termination point due to certain situations, such as the loss of contact with individuals studied or loss of experimental units. This situation leads to the area of progressive censoring.

Wu and Kuş [36] developed a new life test scheme: progressively first-failure censoring scheme, by combining the concept of first failure censoring with progressive censoring. In this scheme, sets with no failures can be removed from the test before the end of the experiment. Based on this scheme, Wu and Kuş [36] derived the maximum likelihood estimators ($MLEs$) and constructed exact and approximate confidence intervals for the parameters of Weibull distribution. Wu and Huang [37] developed the reliability sampling plans for the Weibull distribution. Soliman *et al*. [33-34] derived Bayes and frequentist estimators for the parameters of Gompertz and Burr type XII distributions respectively. Hong *et al*. [16] used the same scheme to construct MLE for the lifetime performance index C_L based on progressively first-failure censoring from Weibull distribution.

The progressive first-failure censoring can be described as follows: Given $m \le n$ and $\mathbf{R} = (R_1, R_2, \ldots, R_m)$ non-negative integers such that $n = m + \sum_{i=1}^{m} R_i$. Let n independent groups with k items within each group be placed on a life testing experiment and only m failures are completely observed. The censoring occurs progressively in m stages. At the time of the first failure $X_{1:m:n:k}$, R_1 random groups and the group with observed failure are randomly removed. Similarly, at the time of the second failure $X_{2:m:n:k}$, R_2 random groups and the group with the second observed failure are randomly removed and so on. Finally, at the time of the *m-th* failure all the remaining active groups (R_m) and the group with the *m-th* observed failure are removed. Then $X_{1:m:n:k} < X_{2:m:n:k} < \cdots < X_{m:m:n:k}$ is the progressive first-failure censored order statistics.

The main advantage of this scheme is that it reduces the time in which more items are used but only m out of $k \times n$ items are observed. Moreover, it includes as special cases, the progressively type-II scheme (when $k = 1$), first-failure scheme (when $\mathbf{R} = (0, 0, \ldots, 0)$), conventional type-II scheme (when $k = 1$ and $\mathbf{R} = (0, 0, \ldots, n - m)$ and the complete sample (when $k = 1, n = m$ and $\mathbf{R} = (0, 0, \ldots, 0)$). Furthermore, the progressively first-failure censored sample $X_{1:m:n:k} < X_{2:m:n:k} < \cdots < X_{m:m:n:k}$ can be considered as a progressively type-II censored sample from a population with distribution function $1 - (1 - F(x))^k$ which enables us to extend all the results on progressive type-II censored order statistics to progressively first-failure censored order statistics.

The Weibull distribution is one of the most popular and widely used models in life testing and reliability theory. The property of having the increasing and decreasing hazard rate functions enables the Weibull distribution to fit lifetime models with monotone hazard rate functions. Nevertheless, it has been found that the Weibull distribution does not provide a satisfactory parametric fit for those lifetime distributions with non-monotone failure rates, such as the unimodal failure rate functions which are common in reliability and biological studies. In this case instead, it is recommended to use a special case of the Weibull distribution, the Inverse Weibull (IW) distribution.

The IW distribution, also known as a type 2 extreme value or the Frechet distribution (Johnson *et al*. [17]), has a long right tail compared to other known distributions. The hazard function of the IW distribution is similar to that of the log-normal and inverse Gaussian distributions (Murthy *et al.* [25]). Carriere [11] used the IW distribution to model the mortality curve of a population. Keller and Kamath [19] suggested that this distribution is a suitable model to describe the degradation phenomena of mechanical components of diesel engines such as pistons, crankshafts, main bearings. Whereas, Erto [13] showed that the IW distribution provided a good fit to several data such as the time to breakdown of an insulating fluid subjected to a constant tension (Nelson [26]).

Let T follow ($\sim$) a two-parameter Weibull distribution (α, β) with probability density function (pdf)

$$f(t;\, \alpha,\, \beta) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} e^{-\left(\frac{\alpha}{t}\right)^{-\beta}},\ t > 0,$$

then the failure time $X = 1/T$ has an IW distribution with pdf

$$f(x;\, \alpha,\, \beta) = \alpha\beta(\alpha x)^{-\beta-1} e^{-(\alpha x)^{-\beta}},\ x > 0, \tag{1}$$

where $\alpha > 0$ and $\beta > 0$, are the scale and shape parameters respectively.

If $X \sim IW(\alpha, \beta)$, then the cumulative distribution function of X is given by:

$$F(x;\, \alpha,\, \beta) = e^{-(\alpha x)^{-\beta}},\ x > 0. \tag{2}$$

Several researches were carried out on IW distribution using classical and Bayesian approaches. For example, Calabria and Pulcini [9] obtained the MLE and the least squares estimates of the parameters of the IW distribution. Calabria and Pulcini [10] considered the Bayesian approach to predict the ordered lifetimes in a future sample when those lifetimes were assumed to follow the IW distribution. Panaitescu *et al.* [29] developed the Bayesian and non-Bayesian analysis in the context of recorded statistic values from a modified IW distribution. Inferential issues for the IW distribution based on censored data are addressed by Maswadah [24], who considered the problem of obtaining the conditional confidence intervals for the parameters and reliability function of IW distribution based on censored generalized order statistics. Mahmoud *et al.* [22] derived the exact expression for the single moment of order statistics from IW distribution. Balakrishnan *et al.* [5] conducted inference on progressive type-II censored data for extreme value distribution. Kundu and Howlader [21] described Bayesian inference and prediction of IW distribution for type-II censored data. They used the Markov Chain Monte Carlo (MCMC) technique to compute the Bayes estimates of the parameters. Kim *et al.* [20] derived the maximum likelihood and the Bayes estimates for the three-parameter exponentiated Weibull distribution for type-II progressively censored sample. As far as we know, the problem of Bayesian estimation using Lindley's approximation based on m progressively first-failure censored samples have not been addressed yet.

The article unfolds as follows: In Section 2 we derive the $MLEs$ of the unknown parameters. The Bayes method is provided in Section 3. In Section 4 we provide the simulation studies, results and conclusion. All methods that are discussed in this article are illustrated in Section 5 through a real life data from highway pavement projects in Amman-Jordan during 2012.

2. Maximum Likelihood Estimator

Suppose that n independent k-unit sets are placed within a test. The ordered m failures are observed under the progressively first-failure.

Let $\mathbf{X} = (X_{1:m:n:k}, X_{2:m:n:k}, \cdots, X_{m:m:n:k})$ with $X_{1:m:n:k} < X_{2:m:n:k} < \cdots < X_{m:m:n:k}$ denote the progressively first-failure censored ordered statistics with the progressive censoring scheme $\mathbf{R}$ from a population with pdf and cdf given in Equations (1) and (2), respectively. For notation simplicity, we will write X_i for $X_{i:m:n:k}$. The likelihood function based on progressively first-failure censored sample (see Wu and Kuş [36]) is given by:

$$L(\alpha, \beta; \mathbf{X}) = A\, k^m \prod_{i=1}^{m} f(x_i; \alpha, \beta)\left[1 - F(x_i; \alpha, \beta)\right]^{k(Ri+1)-1}, \tag{3}$$

where

$$A = n(n-1-R_1)(n-2-R_1-R_2)...(n - \sum_{i=1}^{m}(R_i+1)).$$

In accordance with Equations (1), (2), and (3), the log-likelihood function of α and β based on progressively first-failure censored sample $\mathbf{X}$ becomes

$$\begin{aligned} \ln L(\alpha, \beta; \mathbf{X}) = \text{constant} + m\ln(\alpha\beta) - (\beta+1)\sum_{i=1}^{m}\ln(\alpha x_i) - \sum_{i=1}^{m}(\alpha x_i)^{-\beta} \\ + \sum_{i=1}^{m}\left(k(Ri+1)-1\right)\ln(1 - e^{-(\alpha x_i)^{-\beta}}). \end{aligned} \tag{4}$$

The *MLEs* of the parameters α and β can be obtained by deriving Equation (4) with respect to α and β and equating the normal equations to 0 as follows:

$$\frac{\partial \ln L(\alpha, \beta; \mathbf{X})}{\partial \alpha} = -m + \sum_{i=1}^{m}(\alpha x_i)^{-\beta} + \sum_{i=1}^{m}\frac{\left(k(Ri+1)-1\right)(\alpha x_i)^{-\beta} e^{-(\alpha x_i)^{-\beta}}}{1 - e^{-(\alpha x_i)^{-\beta}}} = 0, \tag{5}$$

$$\begin{aligned} \frac{\partial \ln L(\alpha, \beta; \mathbf{X})}{\partial \beta} = \frac{m}{\beta} + \sum_{i=1}^{m}\ln(\alpha x_i)\left(1 + (\alpha x_i)^{-\beta}\right) \\ + \sum_{i=1}^{m}\frac{\left(k(Ri+1)-1\right)(\alpha x_i)^{-\beta} e^{-(\alpha x_i)^{-\beta}} \ln(\alpha x_i)}{1 - e^{-(\alpha x_i)^{-\beta}}} = 0. \end{aligned} \tag{6}$$

Notice that there are no explicit solutions to Equations (5) and (6). Hence, numerical methods are applied to solve the required equations. For our problem, *EM* algorithm will be used based on Ng *et al.* [27] procedure.

3. Bayesian Estimation

In order to make the optimum decision regarding the Bayesian settings, importance should be given to the choice of the loss function in addition to the choice of the prior distribution. The prior distribution can be used by the researcher who has a degree of certainty about the parameter of his/her model. Here we assume that α and β have independent gamma priors. Gamma priors are very suitable and flexible for α and β in our case, but theoretically, any other log-concave priors can be used.

One of the most popular loss functions is the squared error loss function (*SQR*)

$$L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2, \tag{7}$$

where $\hat{\theta}$ is an estimate of θ. The Bayes estimate under Equation (7) is the posterior mean given by $\hat{\theta}_{SQR} = E_{\pi}\theta.$ The SQR is widely employed in the Bayesian inference due to its computational simplicity. It is a symmetric loss function that gives equal weight to overestimation as well as underestimation. However, this is not a good criteria from a practical point of view. For example, Feynman [14] stated that in the disaster of the space shuttle, Challenger, the management may have overestimated the average life or reliability of solid fuel rocket booster. In estimating reliability and failure rate functions, an overestimation causes more damage than underestimation. To resolve such situation, asymmetrical loss functions are more appropriate to utilize. Varian [38] introduced the *Linex* loss function (Linear-Exponential) in response to the criticism of the SQR and since then it has been widely used by several authors such as Zellner [39], Soliman [32], Helu *et al*. [15], Rastogi and Tripathi [31]. The *Linex* loss function is defined as follows:

$$L(\hat{\theta}, \theta) = \exp(\lambda(\hat{\theta} - \theta)) - \lambda(\hat{\theta} - \theta) - 1, \lambda \neq 0, \tag{8}$$

where $\hat{\theta}$ is an estimate of θ. The magnitude of λ reflects the degree of symmetry while the sign of λ reflects the direction of symmetry. Zellner [39] obtained the Bayesian estimator under (*Linex*) loss function by minimizing the posterior expected loss as follows:

$$\hat{\theta}_{LIN} = -\frac{1}{\lambda} \ln E_{\pi}(e^{-\lambda\theta}), \tag{9}$$

provided that $E_{\pi}(e^{-\lambda\theta})$ exists and finite.

The *Linex* loss function is suitable for situations where overestimation may lead to serious consequences. Moreover, it is known for its flexibility and popularity to estimate the location parameter. On the other hand, Basu and Ebrahimi [8] and Parsian and Farsipour [30] found that the *Linex* loss function is unsuitable for estimating the scale parameter and other quantities. Hence, Basu and Ebrahimi [8] defined a modified *Linex* loss function instead. Later on, Calabria and Pulcini [10] proposed a suitable alternative to the modified *Linex* loss function that is, the general entropy loss function.

The general entropy loss function is defined as

$$L(\hat{\theta}, \theta) = \left(\frac{\hat{\theta}}{\theta}\right)^{\lambda} - \lambda \log\left(\frac{\hat{\theta}}{\theta}\right) - 1, \lambda \neq 0, \tag{10}$$

where λ reflects the magnitude and degree of symmetry. Calabria and Pulcini [10] obtained the Bayesian estimator under Equation (10) as follows:

$$\hat{\theta}_{GE} = \left(E_{\pi}(\theta^{-\lambda})\right)^{\frac{-1}{\lambda}}, \tag{11}$$

provided that $E_{\pi}(\theta^{-\lambda})$ exists and finite.

Norstrom [28] introduced the Precautionary loss function as follows:

$$L(\hat{\theta}, \theta) = \frac{(\theta - \hat{\theta})^2}{\hat{\theta}}, \tag{12}$$

and claimed that it is useful to derive conservative estimators since it approaches infinity near the origin and prevents underestimation. This loss function is suitable for situations where underestimation may lead to serious consequences.

The Bayesian estimator under Equation (12) is given by

$$\hat{\theta}_{PRE} = \sqrt{E_\pi(\theta^2)}, \tag{13}$$

provided that $E_\pi(\theta^2)$ exists and finite.

A common feature of lifetime distributions with a shape parameter is that the Bayes estimators cannot be expressed in closed forms. We suggest using Lindley's approximation to derive the Bayes estimators of the unknown parameters of the IW distribution based on progressively first-failure censored sample.

3.1. Lindley's Approximation Method

It is assumed that α and β have independent gamma priors;

$$\alpha \sim \pi_1(\alpha) = \frac{\alpha^{a-1}e^{-b\alpha}b^a}{\Gamma(a)};\ \beta \sim \pi_2(\beta) = \frac{\beta^{c-1}e^{-c\beta}d^c}{\Gamma(d)}, \tag{14}$$

where a, b, c and d are assumed to be known and non-negative. The joint prior for α and β is $\pi(\alpha, \beta) \propto \alpha^{a-1}e^{-b\alpha}\beta^{c-1}e^{-d\beta}$. Use the same set-ups as in Section 2. If $\underline{x} = (x_1, x_2, ..., x_m)$ is a progressively first-failure censored sample from $IW(\alpha, \beta)$, then the joint posterior pdf of α and β is given by

$$\begin{aligned}\pi(\alpha, \beta \mid \underline{x}) &= \frac{L(\underline{x} \mid \alpha, \beta)\pi(\alpha, \beta)}{\int_0^\infty\int_0^\infty L(\underline{x} \mid \alpha, \beta)\pi(\alpha, \beta)d\alpha d\beta} \\ &= \frac{\alpha^{a-1}\beta^{c-1}e^{-b\alpha}e^{-d\beta}\prod_{i=1}^{m}\alpha\beta(\alpha x)^{-\beta-1}e^{-(\alpha x)^{-\beta}}(1-e^{-(\alpha x)^{-\beta}})^{(k(Ri+1)-1)}}{\int_0^\infty\int_0^\infty \alpha^{a-1}\beta^{c-1}e^{-b\alpha}e^{-d\beta}\prod_{i=1}^{m}\alpha\beta(\alpha x)^{-\beta-1}e^{-(\alpha x)^{-\beta}}(1-e^{-(\alpha x)^{-\beta}})^{(k(Ri+1)-1)}d\alpha d\beta}.\end{aligned} \tag{15}$$

Therefore, the Bayes estimators of any function of α and β say $u(\alpha, \beta)$ are the posterior expected value. Let $u(\alpha, \beta)$ be a function of α and β, then the expected value of $u(\alpha, \beta)$ is given by:

$$\hat{u} = E_\pi(u(\alpha, \beta) \mid \underline{x}) = \frac{\int_0^\infty\int_0^\infty u(\alpha,\beta)e^{\rho(\alpha, \beta)+l(\underline{x}|\alpha, \beta)}d\alpha d\beta}{\int_0^\infty\int_0^\infty e^{\rho(\alpha, \beta)+l(\underline{x}|\alpha, \beta)}d\alpha d\beta}, \tag{16}$$

where, $\rho(\alpha, \beta) = \ln \pi(\alpha, \beta)$ and $l(\underline{x} \mid \alpha, \beta) = \ln L(\underline{x} \mid \alpha, \beta)$.

It can be noticed that $\hat{u}$ is in the form of a ratio of two integrals which can not be simplified to a closed form. Hence Lindley's approximation method is applied to obtain the Bayes estimators of α and β. Then Equation (16) is reduced to the following numerical expression.

$$\begin{aligned}\hat{u} &= u(\hat{\alpha}, \hat{\beta}) + 0.5\Big[(\hat{u}_{\alpha\alpha} + 2\hat{u}_\alpha\hat{\rho}_\alpha)\hat{\sigma}_{\alpha\alpha} + (\hat{u}_{\beta\alpha} + 2\hat{u}_\beta\hat{\rho}_\alpha)\hat{\sigma}_{\beta\alpha} + (\hat{u}_{\alpha\beta} + 2\hat{u}_\alpha\hat{\rho}_\beta)\hat{\sigma}_{\alpha\beta} \\ &+(\hat{u}_{\beta\beta} + 2\hat{u}_\beta\hat{\rho}_\beta)\hat{\sigma}_{\beta\beta}\Big] + 0.5\Big[(\hat{u}_\alpha\hat{\sigma}_{\alpha\alpha} + \hat{u}_\beta\hat{\sigma}_{\alpha\beta})(\hat{l}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha} + \hat{l}_{\alpha\beta\alpha}\hat{\sigma}_{\alpha\beta} + \hat{l}_{\beta\alpha\alpha}\hat{\sigma}_{\beta\alpha} + \hat{l}_{\beta\beta\alpha}\hat{\sigma}_{\beta\beta}) \\ &+ (\hat{u}_\alpha\hat{\sigma}_{\beta\alpha} + \hat{u}_\beta\hat{\sigma}_{\beta\beta})(\hat{l}_{\beta\alpha\alpha}\hat{\sigma}_{\alpha\alpha} + \hat{l}_{\alpha\beta\beta}\hat{\sigma}_{\alpha\beta} + \hat{l}_{\beta\alpha\beta}\hat{\sigma}_{\beta\alpha} + \hat{l}_{\beta\beta\beta}\hat{\sigma}_{\beta\beta})\Big],\end{aligned}$$

where, $\hat{\alpha}$ and $\hat{\beta}$ are the MLE's of α and β respectively, $\hat{u}_{\alpha\alpha} = \left[\partial^2 u(\alpha, \beta) / \partial\alpha\partial\alpha\right]|_{(\hat{\alpha}, \hat{\beta})}$, $\hat{\rho}_\beta = [(a-1)/\hat{\alpha}] - b$, $\hat{\rho}_\beta = \left[(c-1)/\hat{\beta}\right] - d$. Other expressions can be defined similarly with the following definitions.

$$\hat{\sigma}=\begin{bmatrix}\hat{\sigma}_{\alpha\alpha} & \hat{\sigma}_{\alpha\beta}\\ \hat{\sigma}_{\beta\alpha} & \hat{\sigma}_{\beta\beta}\end{bmatrix}=\begin{bmatrix}-\frac{\partial^2 l}{\partial\alpha^2}|_{\alpha=\hat{\alpha},\ \beta=\hat{\beta}} & -\frac{\partial^2 l}{\partial\alpha\partial\beta}|_{\alpha=\hat{\alpha},\ \beta=\hat{\beta}}\\ -\frac{\partial^2 l}{\partial\beta\partial\alpha}|_{\alpha=\hat{\alpha},\ \beta=\hat{\beta}} & -\frac{\partial^2 l}{\partial\beta^2}|_{\alpha=\hat{\alpha},\ \beta=\hat{\beta}}\end{bmatrix},$$

$$\hat{l}_{\alpha\alpha}=\frac{\partial^2 l}{\partial\alpha^2}=\frac{m\hat{\beta}}{\hat{\alpha}^2}-\sum_{i=1}^{m}\frac{(\hat{\beta}+1)\hat{\beta}x_i}{(\hat{\alpha}x_i)^{2+\hat{\beta}}}+\sum_{i=1}^{m}\frac{\left[k(Ri+1)-1\right](\hat{\beta}+1)\hat{\beta}e^{-(\hat{\alpha}x_{i:})^{-\hat{\beta}}}x_i}{(\hat{\alpha}x_i)^{2+\hat{\beta}}\left[1-e^{-(\hat{\alpha}x_{i:})^{-\hat{\beta}}}\right]}$$
$$-\sum_{i=1}^{m}\frac{\left[k(Ri+1)-1\right]\hat{\beta}^2e^{-(\hat{\alpha}x_i)^{-\hat{\beta}}}x_i}{(\hat{\alpha}x_i)^{2+2\hat{\beta}}\left[1-e^{-(\hat{\alpha}x_{i:})^{-\hat{\beta}}}\right]^2},$$

$$\hat{l}_{\alpha\beta}=\frac{\partial^2 l}{\partial\alpha\partial\beta}=\frac{-m}{\hat{\alpha}}-\sum_{i=1}^{m}\frac{x_i\left[\hat{\beta}\log(\hat{\alpha}x_i)-1\right]}{(\hat{\alpha}x_i)^{1+\hat{\beta}}}-\sum_{i=1}^{m}\frac{\left[k(Ri+1)-1\right]x_ie^{-(\hat{\alpha}x_i)^{-\hat{\beta}}}\hat{\beta}\log(\hat{\alpha}x_i)}{(\hat{\alpha}x_i)^{1+2\hat{\beta}}\left[1-e^{-(\hat{\alpha}x_i)^{-\hat{\beta}}}\right]^2}$$
$$+\sum_{i=1}^{m}\frac{\left[k(Ri+1)-1\right]x_ie^{-(\hat{\alpha}x_i)^{-\hat{\beta}}}\left[\hat{\beta}\log(\hat{\alpha}x_i)-1\right]}{(\hat{\alpha}x_i)^{1+\hat{\beta}}\left[1-e^{-(\hat{\alpha}x_i)^{-\hat{\beta}}}\right]},$$

$$\hat{l}_{\alpha\alpha\alpha}=\frac{\partial^3 l}{\partial\alpha^3}=\frac{-2m\hat{\beta}}{\hat{\alpha}^3}+\sum_{i=1}^{m}\frac{(\hat{\beta}+1)\hat{\beta}(\hat{\beta}+2)x_i^3}{(\hat{\alpha}x_i)^{3+\hat{\beta}}}-\sum_{i=1}^{m}\frac{\left[k(Ri+1)-1\right]\hat{\beta}^2(\hat{\beta}+1)e^{-(\hat{\alpha}x_i)^{-\hat{\beta}}}x_i}{(\hat{\alpha}x_i)^{3+2\hat{\beta}}\left[1-e^{-(\hat{\alpha}x_i)^{-\hat{\beta}}}\right]^2}$$
$$-\sum_{i=1}^{m}\frac{e^{-(\hat{\alpha}x_i)^{-\hat{\beta}}R_ix_{i:}^3}\hat{\beta}(\hat{\beta}+1)(\hat{\beta}+2)}{(\hat{\alpha}x_i)^{3+\hat{\beta}}\left[1-e^{-(\hat{\alpha}x_i)^{-\hat{\beta}}}\right]}+\sum_{i=1}^{m}\frac{\left[k(Ri+1)-1\right]\hat{\beta}^3e^{-(\hat{\alpha}x_{i:})^{-\hat{\beta}}}x_i^3\left[1-3e^{-(\hat{\alpha}x_{i:})^{-\hat{\beta}}}\right]}{(\hat{\alpha}x_i)^{3+3\hat{\beta}}\left[1-e^{-(\hat{\alpha}x_i)^{-\hat{\beta}}}\right]^3},$$

$$\hat{l}_{\alpha\alpha\beta}=\frac{\partial^3 l}{\partial\alpha^2\partial\beta}=\frac{m}{\hat{\alpha}^2}-\sum_{i=1}^{m}\frac{x_i\left[1+2\hat{\beta}-\hat{\beta}(\hat{\beta}+1)\log(\hat{\alpha}x_i)\right]}{(\hat{\alpha}x_i)^{2+\hat{\beta}}}$$
$$+\sum_{i=1}^{m}\frac{\left[k(Ri+1)-1\right]x_i\hat{\beta}e^{-(\hat{\alpha}x_i)^{-\hat{\beta}}}\left[2+(1-\hat{\beta})\log(\hat{\alpha}x_i)\right]}{(\hat{\alpha}x_i)^{2+2\hat{\beta}}[1-e^{-(\hat{\alpha}x_i)^{-\hat{\beta}}}]^2}$$
$$+\sum_{i=1}^{m}\frac{\left[k(Ri+1)-1\right]x_i\hat{\beta}^2e^{-(\hat{\alpha}x_i)^{-\hat{\beta}}}\left[1-3e^{-(\hat{\alpha}x_i)^{-\hat{\beta}}}\right]\log(\hat{\alpha}x_i)}{(\hat{\alpha}x_i)^{2+3\hat{\beta}}\left[1-e^{-(\hat{\alpha}x_i)^{-\hat{\beta}}}\right]^3},$$

$$\hat{l}_{\beta\beta}=\frac{\partial^2 l}{\partial\beta^2}=\frac{-m}{\hat{\beta}^2}-\sum_{i=1}^{m}\frac{\log(\hat{\alpha}x_i)^2}{(\hat{\alpha}x_i)^{\hat{\beta}}}+\sum_{i=1}^{m}\frac{\left[k(Ri+1)-1\right]e^{-(\hat{\alpha}x_i)^{-\hat{\beta}}}\log(\hat{\alpha}x_i)^2\left[1-e^{-(\hat{\alpha}x_i)^{-\hat{\beta}}}-(\hat{\alpha}x_i)^{-\hat{\beta}}\right]}{(\hat{\alpha}x_i)^{\hat{\beta}}\left[1-e^{-(\hat{\alpha}x_i)^{-\hat{\beta}}}\right]^2}.$$

3.2. Bayes Estimates under Symmetric Loss Function

- Approximate Bayes estimate of α under squared loss function.

If $u(\alpha, \beta) = \alpha, u_\alpha = 1, u_\beta = u_{\beta\beta} = u_{\alpha\alpha} = u_{\alpha\beta} = u_{\beta\alpha} = 0$, then

$$\hat{\alpha}_{SQR} = \hat{\alpha} + \hat{\rho}_\alpha \hat{\sigma}_{\alpha\alpha} + \hat{\rho}_\beta \hat{\sigma}_{\alpha\beta} + \hat{\sigma}^2_{\beta\alpha} \hat{l}_{\alpha\beta\beta} + \frac{\left[\hat{\sigma}^2_{\alpha\alpha} \hat{l}_{\alpha\alpha\alpha} + 3\hat{\sigma}_{\alpha\alpha}\hat{\sigma}_{\beta\alpha}\hat{l}_{\alpha\beta\alpha} + \hat{\sigma}_{\beta\beta}\hat{\sigma}_{\alpha\alpha}\hat{l}_{\beta\beta\alpha} + \hat{\sigma}_{\beta\alpha}\hat{\sigma}_{\beta\beta}\hat{l}_{\beta\beta\beta}\right]}{2}.$$

- Approximate Bayes estimate of β under squared loss function.

If $u(\alpha, \beta) = \beta, u_\beta = 1, u_{\beta\beta} = u_{\alpha\beta} = u_{\beta\alpha} = u_\alpha = u_{\alpha\alpha} = 0$, then

$$\hat{\beta}_{SQR} = \hat{\beta} + \hat{\rho}_\alpha \hat{\sigma}_{\beta\alpha} + \hat{\rho}_\beta \hat{\sigma}_{\beta\beta} + \hat{\sigma}^2_{\alpha\beta} \hat{l}_{\alpha\beta\alpha} + \frac{\left[\hat{\sigma}^2_{\beta\beta} \hat{l}_{\beta\beta\beta} + 3\hat{\sigma}_{\beta\beta}\hat{\sigma}_{\alpha\beta}\hat{l}_{\alpha\beta\beta} + \hat{\sigma}_{\beta\beta}\hat{\sigma}_{\alpha\alpha}\hat{l}_{\beta\alpha\alpha} + \hat{\sigma}_{\alpha\beta}\hat{\sigma}_{\alpha\alpha}\hat{l}_{\alpha\alpha\alpha}\right]}{2}.$$

3.3. Bayes Estimates Under Asymmetric Loss Functions

(1) The Bayes estimates under *Linex* loss function

- Approximate Bayes estimate of α under *Linex* loss function

If $u(\alpha, \beta) = e^{-\lambda\alpha}, u_\alpha = -\lambda e^{-\lambda\alpha}, u_{\alpha\alpha} = \lambda^2 e^{-\lambda\alpha}, u_\beta = u_{\beta\beta} = u_{\alpha\beta} = u_{\beta\alpha} = 0$, then

$$E_\pi(e^{-\lambda\alpha} \mid \underline{x}) = e^{-\lambda\hat{\alpha}} + \frac{\hat{\mu}_{\alpha\alpha}\hat{\sigma}_{\alpha\alpha}}{2} + \hat{\mu}_\alpha \left[\left(\hat{\rho}_\alpha + \hat{\sigma}_{\beta\alpha}\hat{l}_{\alpha\beta\alpha}\right)\hat{\sigma}_{\alpha\alpha} + \left(\hat{\rho}_\beta + \hat{\sigma}_{\beta\alpha}\hat{l}_{\alpha\beta\beta}\right)\hat{\sigma}_{\alpha\beta}\right] + \frac{\hat{\mu}_\alpha\left[\hat{\sigma}^2_{\alpha\alpha}\hat{l}_{\alpha\alpha\alpha} + \hat{\sigma}_{\alpha\alpha}\hat{\sigma}_{\beta\beta}\hat{l}_{\beta\beta\alpha} + \hat{\sigma}_{\beta\alpha}\hat{\sigma}_{\alpha\alpha}\hat{l}_{\beta\alpha\alpha} + \hat{\sigma}_{\beta\alpha}\hat{\sigma}_{\beta\beta}\hat{l}_{\beta\beta\beta}\right]}{2}.$$

Hence, the Bayes estimate of α is obtained by

$$\hat{\alpha}_{LIN} = -\frac{1}{\lambda}\ln E_\pi(e^{-\lambda\alpha} \mid \underline{x}).$$

- Approximate Bayes estimate of β under *Linex* loss function

If $u(\alpha, \beta) = e^{-\lambda\beta}, u_\beta = -\lambda e^{-\lambda\beta}, u_{\beta\beta} = \lambda^2 e^{-\lambda\beta}, u_\alpha = u_{\alpha\alpha} = u_{\alpha\beta} = u_{\beta\alpha} = 0$, then

$$E_\pi(e^{-\lambda\beta} \mid \underline{x}) = e^{-\lambda\hat{\beta}} + \frac{\hat{u}_\beta\hat{\sigma}_{\beta\beta}\left(\hat{\sigma}_{\beta\beta}\hat{l}_{\beta\beta\beta} + 3\hat{\sigma}_{\alpha\beta}\hat{l}_{\beta\beta\alpha} + \hat{\sigma}_{\alpha\alpha}\hat{l}_{\beta\alpha\alpha}\right)}{2} + \frac{\left(\hat{u}_{\beta\beta} + 2\hat{u}_\beta\hat{\rho}_\beta\right)\hat{\sigma}_{\beta\beta} + \hat{u}_\beta\hat{\sigma}_{\alpha\beta}\hat{\sigma}_{\alpha\alpha}\hat{l}_{\alpha\alpha\alpha}}{2} + \hat{u}_\beta\left(\hat{\rho}_\alpha + \hat{\sigma}_{\alpha\beta}\hat{l}_{\alpha\beta\alpha}\right)\hat{\sigma}_{\alpha\beta}.$$

Hence, the Bayes estimate of β is obtained by

$$\hat{\beta}_{LIN} = -\frac{1}{\lambda}\ln E_\pi(e^{-\lambda\beta} \mid \underline{x}).$$

(2) The Bayes estimates under general entropy loss function

- Approximate Bayes estimate of α under general entropy

If $u(\alpha, \beta) = \alpha^{-\lambda}, u_\alpha = -\lambda / \alpha^{\lambda+1}, u_{\alpha\alpha} = \lambda(\lambda+1)/\alpha^{\lambda+2}, u_\beta = u_{\beta\beta} = u_{\alpha\beta} = u_{\beta\alpha} = 0$, then

$$E_\pi(\alpha^{-\lambda}\,|\,\underline{x})=\hat{\alpha}^{-\lambda}+\frac{(\hat{u}_{\alpha\alpha}+2\hat{u}_\alpha\hat{\rho}_\alpha)\hat{\sigma}_{\alpha\alpha}}{2}+\hat{u}_\alpha\left(\hat{\rho}_\beta\hat{\sigma}_{\alpha\beta}+\hat{\sigma}_{\alpha\alpha}\hat{\sigma}_{\beta\alpha}\hat{l}_{\alpha\beta\alpha}+\hat{\sigma}^2_{\beta\alpha}\hat{l}_{\alpha\beta\beta}\right)$$
$$+\frac{\hat{u}_\alpha\left[\hat{\sigma}^2_{\alpha\alpha}\hat{l}_{\alpha\alpha\alpha}+\hat{\sigma}_{\alpha\alpha}\left(\hat{\sigma}_{\beta\beta}\hat{l}_{\beta\beta\alpha}+\hat{\sigma}_{\beta\alpha}\hat{l}_{\beta\alpha\alpha}\right)+\hat{\sigma}_{\beta\alpha}\hat{\sigma}_{\beta\beta}\hat{l}_{\beta\beta\beta}\right]}{2}.$$

So, the Bayes estimate of α is $\hat{\alpha}_{GE}=[E_\pi(\alpha^{-\lambda}\,|\,\underline{x})]^{-1/\lambda}$.

- Approximate Bayes estimator of β under the general entropy.

If $u(\alpha,\beta)=\beta^{-\lambda}$, $u_\beta=-\lambda/\beta^{\lambda+1}$, $u_{\beta\beta}=\lambda(\lambda+1)/\beta^{\lambda+2}$, $u_\alpha=u_{\alpha\alpha}=u_{\alpha\beta}=u_{\beta\alpha}=0$, then

$$E_\pi(\beta^{-\lambda}\,|\,\underline{x})=\hat{\beta}^{-\lambda}+\hat{u}_\beta\left(\hat{\rho}_\alpha\hat{\sigma}_{\beta\alpha}+\hat{\rho}_\beta\hat{\sigma}_{\beta\beta}\right)+\frac{\hat{u}_{\beta\beta}\hat{\sigma}_{\beta\beta}+\hat{u}_\beta\hat{\sigma}_{\alpha\beta}\hat{\sigma}_{\alpha\alpha}\hat{l}_{\alpha\alpha\alpha}}{2}$$
$$+\frac{\hat{u}_\beta\left[\left(\hat{\sigma}_{\beta\beta}\hat{l}_{\beta\beta\beta}+\hat{\sigma}_{\alpha\alpha}\hat{l}_{\beta\alpha\alpha}\right)\hat{\sigma}_{\beta\beta}+\left(2\hat{\sigma}_{\alpha\beta}\hat{l}_{\alpha\beta\alpha}+3\hat{\sigma}_{\beta\beta}\hat{l}_{\beta\beta\alpha}\right)\hat{\sigma}_{\alpha\beta}\right]}{2}.$$

Hence, $\hat{\beta}_{GE}=[E_\pi(\beta^{-\lambda}\,|\,\underline{x})]^{-1/\lambda}$.

(3) The Bayes estimates under Precautionary loss function

- Approximate Bayes estimate of α under the Precautionary loss function

If $u(\alpha,\beta)=\alpha^2$, $u_\alpha=2\alpha$, $u_{\alpha\alpha}=2$, $u_\beta=u_{\beta\beta}=u_{\alpha\beta}=u_{\beta\alpha}=0$, then

$$E_\pi(\alpha^2\,|\,\underline{x})=\hat{\alpha}^2+\hat{u}_\alpha\hat{\rho}_\beta\hat{\sigma}_{\alpha\beta}+\frac{\left[(\hat{u}_{\alpha\alpha}+2\hat{u}_\alpha\hat{\rho}_\alpha)\hat{\sigma}_{\alpha\alpha}+\hat{u}_\alpha\left(\hat{\sigma}^2_{\alpha\alpha}\hat{l}_{\alpha\alpha\alpha}+\hat{\sigma}_{\beta\alpha}\hat{l}_{\alpha\beta\alpha}\right)\right]}{2}$$
$$+\frac{\hat{u}_\alpha\left[\hat{\sigma}_{\alpha\alpha}\left(\hat{\sigma}_{\beta\beta}\hat{l}_{\beta\beta\alpha}+\hat{\sigma}_{\beta\alpha}\hat{l}_{\beta\alpha\alpha}\right)+\hat{\sigma}_{\beta\alpha}\left(2\hat{\sigma}_{\beta\alpha}\hat{l}_{\alpha\beta\beta}+\hat{\sigma}_{\beta\beta}\hat{l}_{\beta\beta\beta}\right)\right]}{2}.$$

Therefore, $\hat{\alpha}_{PRE}=\sqrt{E_\pi(\alpha^2\,|\,\underline{x})}$.

- Approximate Bayes estimate of β under the Precautionary loss function

If $u(\alpha,\beta)=\beta^2$, $u_\beta=2\beta$, $u_{\beta\beta}=2u_\alpha=u_{\alpha\alpha}=u_{\alpha\beta}=u_{\beta\alpha}=0$, then

$$E_\pi(\beta^2\,|\,\underline{x})=\hat{\beta}^2+\hat{u}_\beta\hat{\rho}_\alpha\hat{\sigma}_{\beta\alpha}+\frac{\left[(\hat{u}_{\beta\beta}+2\hat{u}_\beta\hat{\rho}_\beta)\hat{\sigma}_{\beta\beta}+\hat{\mu}_\beta\hat{\sigma}_{\alpha\beta}\hat{\sigma}_{\alpha\alpha}\hat{l}\hat{\sigma}^2_{\beta\beta}\hat{l}_{\beta\beta\beta}\right]}{2}$$
$$+\frac{\hat{u}_\beta\left[2\hat{\sigma}^2_{\alpha\beta}\hat{l}_{\alpha\beta\alpha}+\hat{\sigma}_{\beta\beta}\hat{\sigma}_{\alpha\alpha}\hat{l}_{\beta\alpha\alpha}+3\hat{\sigma}_{\alpha\beta}\hat{\sigma}_{\beta\beta}\hat{l}_{\beta\beta\alpha}\right]}{2}.$$

So, $\hat{\beta}_{PRE}=\sqrt{E_\pi(\beta^2\,|\,\underline{x})}$.

4. Simulation Study

The purpose of the simulation study is to compare the performance of the *MLE* and the Bayesian estimates based on symmetric and asymmetric loss functions using independent gamma priors for the shape and scale parameters.

Values of α and β are generated from π_1 and π_2 given in Equation (14) with specified parameters "a", "b", "c", and "d". 7000 progressively first failure censored samples are randomly simulated using $a=0.3, 0.5, 1.5, 3.0$, $b=1$, $c=0.5, 1.5, 2.5, 3.0$, $d=1$, and different combinations of n, m, k and different censoring schemes $\mathbf{R} = (R_1, ..., R_m)$. The data were simulated using Balakrishnan and Aggarwala [3] algorithm based on the fact that progressively first-failure censored sample with distribution $F(x)$ can be viewed as a progressively type-II censored sample from a population with distribution function $1-(1-F(x))^k$.

We obtain the *MLEs* of α and β by solving the nonlinear Equations (5) and (6) using *EM* algorithm. The criteria used for comparing all the above estimators are the Bias and the mean squared error (*MSE*).

Suppose $\hat{\theta}_i$ $(=\hat{\alpha}_i, \hat{\beta}_i)$ is the estimate of θ $(=\alpha, \beta)$ for the i-*th* simulated data set, then the *Bias* and *MSE* are computed as follows:

(i) $Bias = \dfrac{1}{7000} \sum_{i=1}^{7000} \left| \hat{\theta}_i - \theta \right|.$

(ii) $MSE = \dfrac{1}{7000} \sum_{i=1}^{7000} (\hat{\theta}_i - \theta)^2.$

5. Data Analysis and Comparison Study

Due to the large number of tables and results, only results for (*a*=0.5, *c*=3.0) and (*a*=1.5, *c*=2.5) are reported. Results are summarized in Tables 1-8 provided at the end of this section as follows:

- Tables 1-4 display the *Bias* and the *MSE* values for the estimates of α.
- Tables 5-8 display the *Bias* and the *MSE* values for the estimates of β.

Throughout this section we will refer to the scheme $\mathbf{R} = (n-m, 0 ..., 0)$ by case-I, $\mathbf{R} = (0, ..., 0, n-m)$ by case-II. As for the schemes that are between these two extremes, we will call them case-III.

A summary of the results is provided below:

- For progressively first-failure censoring ($k = 3 \,\&\, 5$) we can easily notice that case-I is the most efficient scheme in terms of *Bias* and *MSE* values for both estimates of α and β.
- On the other hand, when $k = 1$ which indicates the progressive type-II censoring, we find that case-II is the most efficient in terms of *Bias* and *MSE* values for estimating α.

However, in case of β, the scheme that provides minimum *Bias* and minimum *MSE* values will depend on the choice of the effective size m i.e. for fixed value of n and small values of m, case-I is the most efficient scheme while for larger values of m, case-II overcomes case-I in terms of *Bias* and *MSE* values.

- It is important to point out the following:
  - (i) In the case of efficient schemes, the Bayes estimates outperform the *MLE* for both estimates of α and β, however, they are equivalent for the remaining schemes.
  - (ii) The Bayes estimates based on asymmetric loss functions are equivalent and slightly better than the estimates based on squared error loss function.
  - (iii) The *Bias* and *MSE* values decrease as the effective sample proportion m/n increases for fixed $k \,\&\, n$ and for all estimates of α and β.

- When $\lambda < |0.5|$, the estimates based on *Linex* and squared error loss functions are equivalent.
- When $\lambda = -1$, $\hat{\beta}_{GE}$ has poor performance in terms of *Bias*.

Table 1. Bias of $\hat{\alpha}(.)$ when $(a, b) = (0.5, 3.0)$.

k n m	Scheme	$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{SQR}$	$\hat{\alpha}_{PRE}$	$\hat{\alpha}_{LIN}$		$\hat{\alpha}_{GE}$	
					λ=-0.5	λ=0.5	λ=-0.5	λ=0.5
1 20 10	(10, 0, ..., 0)	0.0506	0.0516	0.0846	0.0613	0.0418	0.0590	0.0358
	(0, ..., 0, 10)	0.0380	0.0117	0.0243	0.0025	0.0206	0.0084	0.0160
	(2, 0, 2, 0, 2, 0, 0, 2, 0, 2)	0.0406	0.0170	0.0459	0.0056	0.0217	0.0224	0.0175
15	(5, 0, ..., 0)	0.0255	0.0384	0.0824	0.0531	0.0243	0.0445	0.0279
	(0, ..., 0, 5)	0.0186	0.0033	0.0165	0.0020	0.0087	0.0017	0.0074
	(0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1)	0.0218	0.0139	0.0232	0.0043	0.0108	0.0064	0.0155
50 20	(30, 0, ..., 0)	0.0315	0.0245	0.0427	0.0293	0.0196	0.0281	0.0230
	(0, ..., 0, 30)	0.0253	0.0039	0.0203	0.0076	0.0016	0.0123	0.0171
	(3, 0, 3, 0, ..., 3, 0)	0.0262	0.0081	0.0235	0.0122	0.0092	0.0175	0.0185
30	(20, 0, ..., 0)	0.0151	0.0169	0.0416	0.0243	0.0098	0.0220	0.0157
	(0,...,0,20)	0.0111	0.0028	0.0063	0.0028	0.0002	0.0020	0.0047
	(2,0,0,2,0,0,...,2,0,0)	0.0129	0.0060	0.0177	0.0075	0.0041	0.0072	0.0087
3 20 10	(10,0,...,0)	0.0639	0.0568	0.0333	0.0500	0.0619	0.0467	0.0233
	(0, ..., 0, 10)	0.0849	0.2209	0.2329	0.2241	0.2170	0.2656	0.1384
	(2, 0, 2, 0, 2, 0, 0, 2, 0, 2)	0.0755	0.0906	0.1192	0.0990	0.0820	0.1087	0.0804
15	(5, 0, ..., 0)	0.0390	0.0037	0.0179	0.0020	0.0092	0.0027	0.0213
	(0, ..., 0, 5)	0.0429	0.0445	0.0642	0.0500	0.0390	0.0512	0.0419
	(0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1)	0.0421	0.0288	0.0498	0.0346	0.0231	0.0355	0.0362
50 20	(30, 0, ..., 0)	0.0325	0.0445	0.0314	0.0412	0.0475	0.0366	0.0097
	(0, ..., 0, 30)	0.0478	0.1622	0.1669	0.1633	0.1609	0.1806	0.0878
	(3, 0, 3, 0, ..., 3, 0)	0.0414	0.0574	0.0731	0.0618	0.0529	0.0651	0.0463
30	(20, 0, ..., 0)	0.0193	0.0095	0.0014	0.0067	0.0123	0.0053	0.0075
	(0,...,0,20)	0.0245	0.0452	0.0548	0.0478	0.0425	0.0482	0.0303
	(2,0,0,2,0,0,...,2,0,0)	0.0230	0.0228	0.0336	0.0257	0.0199	0.0261	0.0222
5 20 10	(10,0,...,0)	0.0728	0.0688	0.0597	0.0649	0.0720	0.0535	0.0362
	(0, ..., 0, 10)	0.1086	0.4025	0.3785	0.3881	0.4209	0.5546	0.2557
	(2, 0, 2, 0, 2, 0, 0, 2, 0, 2)	0.0934	0.1529	0.1778	0.1600	0.1453	0.1971	0.1220
15	(5, 0, ..., 0)	0.0469	0.0166	0.0377	0.0223	0.0111	0.0249	0.0257
	(0, ..., 0, 5)	0.0556	0.1197	0.1354	0.1242	0.1147	0.1355	0.0778
	(0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1)	0.0526	0.0713	0.0914	0.0771	0.0655	0.0828	0.0587
50 20	(30, 0, ..., 0)	0.0349	0.0556	0.0470	0.0533	0.0543	0.0457	0.0137
	(0, ..., 0, 30)	0.0589	0.2297	0.2256	0.2267	0.2331	0.2756	0.1277
	(3, 0, 3, 0, ..., 3, 0)	0.0489	0.0590	0.0758	0.0637	0.0576	0.0724	0.0540
30	(20, 0, ..., 0)	0.0224	0.0043	0.0062	0.0015	0.0069	0.0002	0.0066
	(0, ..., 0, 20)	0.0313	0.0926	0.1001	0.0947	0.0903	0.0988	0.0510
	(2, 0, 0, 2, 0, 0, ..., 2, 0, 0)	0.0280	0.0406	0.0516	0.0436	0.0376	0.0451	0.0317

Table 2. *MSL* of $\hat{\alpha}(.)$ when $(a, b) = (0.5, 3.0)$.

k n m	Scheme	$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{SQR}$	$\hat{\alpha}_{PRE}$	$\hat{\alpha}_{LIN}$		$\hat{\alpha}_{GE}$	
					λ=-0.5	λ=0.5	λ=-0.5	λ=0.5
1 20 10	(10, 0, ..., 0)	0.0606	0.0447	0.0499	0.0487	0.0414	0.0442	0.0486
	(0, ..., 0, 10)	0.0405	0.0319	0.0329	0.0332	0.0309	0.0280	0.0334
	(2, 0, 2, 0, 2, 0, 0, 2, 0, 2)	0.0454	0.0344	0.0358	0.0360	0.0330	0.0300	0.0365
15	(5, 0, ..., 0)	0.0387	0.0340	0.0377	0.0364	0.0318	0.0389	0.0362
	(0, ..., 0, 5)	0.0315	0.0252	0.0270	0.0263	0.0241	0.0246	0.0272
	(0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1)	0.0336	0.0280	0.0303	0.0296	0.0266	0.0289	0.0301
50 20	(30, 0, ..., 0)	0.0290	0.0244	0.0258	0.0255	0.0234	0.0243	0.0259
	(0, ..., 0, 30)	0.0176	0.0165	0.0166	0.0168	0.0162	0.0161	0.0166
	(3, 0, 3, 0, ..., 3, 0)	0.0199	0.0180	0.0183	0.0184	0.0176	0.0173	0.0183
30	(20, 0, ..., 0)	0.0192	0.0178	0.0186	0.0184	0.0172	0.0189	0.0185
	(0, ..., 0, 20)	0.0135	0.0121	0.0124	0.0124	0.0119	0.0118	0.0126
	(2, 0, 0, 2, 0, 0, ..., 2, 0, 0)	0.0151	0.0138	0.0141	0.0141	0.0135	0.0138	0.0143
3 20 10	(10, 0, ..., 0)	0.0639	0.0568	0.0383	0.0388	0.0369	0.0273	0.0340
	(0, ..., 0, 10)	0.0849	0.2209	0.0654	0.0681	0.0693	0.1121	0.0640
	(2, 0, 2, 0, 2, 0, 0, 2, 0, 2)	0.0755	0.0906	0.0523	0.0543	0.0522	0.0613	0.0468

Table 2. (Continued).

k	n	m	Scheme	$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{SQR}$	$\hat{\alpha}_{PRE}$	$\hat{\alpha}_{LIN}$ λ =-0.5	$\hat{\alpha}_{LIN}$ λ =0.5	$\hat{\alpha}_{GE}$ λ =-0.5	$\hat{\alpha}_{GE}$ λ =0.5
3	20	15	(5, 0, ..., 0)	0.0390	0.0037	0.0243	0.0248	0.0237	0.0221	0.0236
			(0, ..., 0, 5)	0.0429	0.0445	0.0261	0.0266	0.0257	0.0273	0.0249
			(0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1)	0.0421	0.0288	0.0261	0.0267	0.0256	0.0262	0.0249
	50	20	(30, 0, ..., 0)	0.0325	0.0445	0.0174	0.0175	0.0169	0.0140	0.0165
			(0, ..., 0, 30)	0.0478	0.1622	0.0311	0.0321	0.0329	0.0459	0.0289
			(3, 0, 3, 0, ..., 3, 0)	0.0414	0.0574	0.0336	0.0244	0.0238	0.0262	0.0213
		30	(20, 0, ..., 0)	0.0193	0.0095	0.0119	0.0120	0.0117	0.0111	0.0117
			(0, ..., 0, 20)	0.0245	0.0452	0.0132	0.0134	0.0132	0.0143	0.0127
			(2, 0, 0, 2, 0, 0, ..., 2, 0, 0)	0.0230	0.0228	0.0211	0.0130	0.0128	0.0131	0.0124
5	20	10	(10, 0, ..., 0)	0.0728	0.0688	0.0597	0.0649	0.0720	0.0535	0.0362
			(0, ..., 0, 10)	0.1086	0.4025	0.3785	0.3881	0.4209	0.5546	0.2557
			(2, 0, 2, 0, 2, 0, 0, 2, 0, 2)	0.0934	0.1529	0.1778	0.1600	0.1453	0.1971	0.1220
		15	(5, 0, ..., 0)	0.0469	0.0166	0.0377	0.0223	0.0111	0.0249	0.0257
			(0, ..., 0, 5)	0.0556	0.1197	0.1354	0.1242	0.1147	0.1355	0.0778
			(0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1)	0.0526	0.0713	0.0914	0.0771	0.0655	0.0828	0.0587
	50	20	(30, 0, ..., 0)	0.0349	0.0556	0.0470	0.0533	0.0543	0.0457	0.0137
			(0, ..., 0, 30)	0.0589	0.2297	0.2256	0.2267	0.2331	0.2756	0.1277
			(3, 0, 3, 0, ..., 3, 0)	0.0489	0.0590	0.0758	0.0637	0.0576	0.0724	0.0540
		30	(20, 0, ..., 0)	0.0224	0.0043	0.0062	0.0015	0.0069	0.0002	0.0066
			(0, ..., 0, 20)	0.0313	0.0926	0.1001	0.0947	0.0903	0.0988	0.0510
			(2, 0, 0, 2, 0, 0, ..., 2, 0, 0)	0.0280	0.0406	0.0516	0.0436	0.0376	0.0451	0.0317

Table 3. Bias of $\hat{\alpha}(.)$ when $(a, b) = (1.5, 2.5)$.

k	n	m	Scheme	$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{SQR}$	$\hat{\alpha}_{PRE}$	$\hat{\alpha}_{LIN}$ λ =-0.5	$\hat{\alpha}_{LIN}$ λ =0.5	$\hat{\alpha}_{GE}$ λ =-0.5	$\hat{\alpha}_{GE}$ λ =0.5
1	20	10	(10, 0, ..., 0)	0.0240	0.0254	0.0427	0.0339	0.0183	0.0506	0.0206
			(0, ..., 0, 10)	0.0186	0.0102	0.0058	0.0024	0.0169	0.0156	0.0124
			(2, 0, 2, 0, 2, 0, 0, 2, 0, 2)	0.0195	0.0091	0.0194	0.0130	0.0174	0.0178	0.0130
		15	(5, 0, ..., 0)	0.0103	0.0203	0.0406	0.0330	0.0086	0.0385	0.0110
			(0, ..., 0, 5)	0.0068	0.0075	0.0043	0.0002	0.0016	0.0057	0.0032
			(0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1)	0.0086	0.0053	0.0076	0.0004	0.0139	0.0148	0.0066
	50	20	(30, 0, ..., 0)	0.0161	0.0113	0.0199	0.0158	0.0071	0.0220	0.0135
			(0, ..., 0, 30)	0.0140	0.0055	0.0115	0.0075	0.0043	0.0029	0.0117
			(3, 0, 3, 0, ..., 3, 0)	0.0142	0.0071	0.0142	0.0100	0.0083	0.0084	0.0122
	50	30	(20, 0, ..., 0)	0.0066	0.0076	0.0197	0.0143	0.0029	0.0125	0.0066
			(0, ..., 0, 20)	0.0050	0.0031	0.0024	0.0024	0.0025	0.0027	0.0029
			(2, 0, 0, 2, 0, 0, ..., 2, 0, 0)	0.0059	0.0069	0.0073	0.0079	0.0042	0.0074	0.0045
3	20	10	(10, 0, ..., 0)	0.0358	0.0206	0.0065	0.0134	0.0268	0.0470	0.0250
			(0, ..., 0, 10)	0.0499	0.1788	0.1779	0.1769	0.1786	0.2913	0.0728
			(2, 0, 2, 0, 2, 0, 0, 2, 0, 2)	0.0439	0.0859	0.0977	0.0919	0.0785	0.1322	0.0503
		15	(5, 0, ..., 0)	0.0218	0.0040	0.0145	0.0094	0.0011	0.0010	0.0180
			(0, ..., 0, 5)	0.0247	0.0386	0.0476	0.0432	0.0335	0.0569	0.0262
			(0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1)	0.0240	0.0282	0.0381	0.0333	0.0230	0.0403	0.0239
	50	20	(30, 0, ..., 0)	0.0186	0.0211	0.0139	0.0175	0.0243	0.0413	0.0115
			(0, ..., 0, 30)	0.0287	0.1397	0.1369	0.1371	0.1409	0.2260	0.0473
			(3, 0, 3, 0, ..., 3, 0)	0.0246	0.0604	0.0667	0.0634	0.0565	0.0929	0.0336
3	50	30	(20,0,...,0)	0.0108	0.0028	0.0025	0.0001	0.0054	0.0072	0.0082
			(0,...,0,20)	0.0145	0.0389	0.0431	0.0409	0.0364	0.0596	0.0180
			(2,0,0,2,0,0,...,2,0,0)	0.0134	0.0228	0.0278	0.0253	0.0253	0.0336	0.0300
5	20	10	(10, 0, ..., 0)	0.0419	0.0283	0.0384	0.0334	0.0229	0.0388	0.0318
			(0, ..., 0, 10)	0.0642	0.3514	0.3129	0.3211	0.3910	0.6000	0.1234
			(2, 0, 2, 0, 2, 0, 0, 2, 0, 2)	0.0549	0.1618	0.1663	0.1626	0.1582	0.2636	0.0747
		15	(5, 0, ..., 0)	0.0272	0.0118	0.0010	0.0054	0.0180	0.0358	0.0267
			(0, ..., 0, 5)	0.0329	0.1059	0.1105	0.1077	0.1025	0.1682	0.0448
			(0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1)	0.0309	0.0714	0.0794	0.0753	0.0663	0.1103	0.0371
	50	20	(30, 0, ..., 0)	0.0204	0.0202	0.0138	0.0171	0.0232	0.0412	0.0133
			(0, ..., 0, 30)	0.0353	0.2307	0.2117	0.2164	0.2454	0.3832	0.0709
			(3, 0, 3, 0, ..., 3, 0)	0.0293	0.0875	0.0925	0.0893	0.0839	0.1380	0.0389
	50	30	(20, 0, ..., 0)	0.0130	0.0063	0.0115	0.0088	0.0036	0.0062	0.0116
			(0, ..., 0, 20)	0.0188	0.0855	0.0868	0.0856	0.0842	0.1362	0.0294
			(2, 0, 0, 2, 0, 0, ..., 2, 0, 0)	0.0167	0.0454	0.0498	0.0474	0.0426	0.0698	0.0209

Table 4. MSL of $\hat{\alpha}(.)$ when $(a, b) = (1.5, 2.5)$.

k n m	Scheme	$\hat{\alpha}_{\text{MLE}}$	$\hat{\alpha}_{\text{SQR}}$	$\hat{\alpha}_{\text{PRE}}$	$\hat{\alpha}_{\text{LIN}}$		$\hat{\alpha}_{\text{GE}}$	
					λ =-0.5	λ =0.5	λ =-0.5	λ =0.5
1 20 10	(10, 0, ..., 0)	0.0259	0.0233	0.0243	0.0249	0.0218	0.0251	0.0245
	(0, ..., 0, 10)	0.0178	0.0161	0.0162	0.0165	0.0155	0.0144	0.0167
	(2, 0, 2, 0, 2, 0, 0, 2, 0, 2)	0.0198	0.0177	0.0179	0.0183	0.0170	0.0159	0.0185
15	(5, 0, ..., 0)	0.0173	0.0167	0.0174	0.0175	0.0158	0.0196	0.0170
	(0, ..., 0, 5)	0.0143	0.0130	0.0133	0.0134	0.0124	0.0126	0.0136
	(0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1)	0.0152	0.0141	0.0145	0.0147	0.0135	0.0147	0.0146
50 20	(30, 0, ..., 0)	0.0129	0.0120	0.0122	0.0124	0.0116	0.0120	0.0125
	(0, ..., 0, 30)	0.0080	0.0078	0.0078	0.0078	0.0076	0.0077	0.0078
	(3, 0, 3, 0, ..., 3, 0)	0.0090	0.0086	0.0086	0.0087	0.0085	0.0084	0.0087
50 30	(20, 0, ..., 0)	0.0087	0.0085	0.0086	0.0087	0.0082	0.0090	0.0086
	(0, ..., 0, 20)	0.0062	0.0059	0.0060	0.0060	0.0058	0.0057	0.0060
	(2, 0, 0, 2, 0, 0, ..., 2, 0, 0)	0.0069	0.0067	0.0067	0.0067	0.0065	0.0066	0.0068
3 20 10	(10, 0, ..., 0)	0.0182	0.0183	0.0180	0.0186	0.0179	0.0156	0.0174
	(0, ..., 0, 10)	0.0187	0.0241	0.0233	0.0235	0.0246	0.0423	0.0217
	(2, 0, 2, 0, 2, 0, 0, 2, 0, 2)	0.0185	0.0212	0.0207	0.0213	0.0210	0.0282	0.0191
15	(5, 0, ..., 0)	0.0110	01150	0.0114	0.0116	0.0113	0.0111	0.0111
	(0, ..., 0, 5)	0.0114	0.0115	0.0114	0.0116	0.0113	0.0127	0.0112
	(0,0,1,0,1,1,0,0,0,1,0,0,0,0,1)	0.0117	0.0118	0.0116	0.0119	0.0116	0.0128	0.0113
50 20	(30,0,...,0)	0.0088	0.0085	0.0085	0.0086	0.0084	0.0075	0.0083
	(0,...,0,30)	0.0093	0.0122	0.0119	0.0119	0.0124	0.0197	0.0107
	(3,0,3,0,...,3,0)	0.0087	0.0102	0.0100	0.0101	0.0101	0.0128	0.0092
50 30	(20,0,...,0)	0.0057	0.0057	0.0056	0.0057	0.0056	0.0055	0.0056
	(0,...,0,20)	0.0055	0.0058	0.0058	0.0058	0.0058	0.0067	0.0056
	(2,0,0,2,0,0,...,2,0,0)	0.0056	0.0059	0.0058	0.0059	0.0059	0.0063	0.0063
5 20 10	(10,0,...,0)	0.0185	0.0208	0.0208	0.0212	0.0203	0.0204	0.0177
	(0,...,0,10)	0.0227	0.0418	0.0381	0.0372	0.0508	0.1109	0.0334
	(2,0,2,0,2,0,0,2,0,2)	0.0205	0.0305	0.0288	0.0296	0.0314	0.0539	0.0241
15	(5,0,...,0)	0.0116	0.0127	0.0124	0.0127	0.0125	0.0139	0.0117
	(0,...,0,5)	0.0121	0.0147	0.0144	0.0146	0.0148	0.0212	0.0133
	(0,0,1,0,1,1,0,0,0,1,0,0,0,0,1)	0.0120	0.0141	0.0138	0.0141	0.0140	0.0183	0.0127
50 20	(30,0,...,0)	0.0084	0.0088	0.0088	0.0089	0.0087	0.0085	0.0080
	(0,...,0,30)	0.0114	0.0204	0.0188	0.0188	0.0228	0.0432	0.0153
	(3,0,3,0,...,3,0)	0.0098	0.0137	0.0132	0.0135	0.0138	0.0200	0.0109
50 30	(20,0,...,0)	0.0056	0.0058	0.0058	0.0058	0.0057	0.0060	0.0055
	(0,...,0,20)	0.0063	0.0076	0.0075	0.0075	0.0076	0.0105	0.0068
	(2,0,0,2,0,0,...,2,0,0)	0.0060	0.0068	0.0068	0.0068	0.0068	0.0082	0.0062

Table 5. Bias of $\hat{\beta}(.)$ when $(a, b) = (0.5, 3.0)$.

k n m	Scheme	$\hat{\beta}_{\text{MLE}}$	$\hat{\beta}_{\text{SQR}}$	$\hat{\beta}_{\text{PRE}}$	$\hat{\beta}_{\text{LIN}}$		$\hat{\beta}_{\text{GE}}$	
					λ =-0.5	λ =0.5	λ =-0.5	λ =0.5
1 20 10	(10, 0, ..., 0)	0.0901	0.0161	0.0357	0.0240	0.0100	0.0060	0.0485
	(0, ..., 0, 10)	0.1208	0.0342	0.0550	0.0428	0.0266	0.0068	0.0703
	(2, 0, 2, 0, 2, 0, 0, 2, 0, 2)	0.1081	0.0212	0.0409	0.0292	0.0141	0.0164	0.0590
15	(5, 0, ..., 0)	0.0700	0.0151	0.0292	0.0205	0.0090	0.0022	0.0373
	(0, ..., 0, 5)	0.0633	0.0006	0.0120	0.0041	0.0050	0.0001	0.0338
	(0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1)	0.0687	0.0118	0.0259	0.0172	0.0068	0.0037	0.0386
50 20	(30, 0, ..., 0)	0.0377	0.0050	0.0116	0.0074	0.0041	0.0067	0.0198
	(0, ..., 0, 30)	0.0552	0.0335	0.0443	0.0376	0.0294	0.0325	0.0432
	(3, 0, 3, 0, ..., 3, 0)	0.0486	0.0202	0.0302	0.0240	0.0166	0.0173	0.0340
50 30	(20, 0, ..., 0)	0.0335	0.0047	0.0112	0.0071	0.0027	0.0022	0.0188
	(0, ..., 0, 20)	0.0270	0.0010	0.0078	0.0022	0.0023	0.0011	0.0161
	(2, 0, 0, 2, 0, 0, ..., 2, 0, 0)	0.0318	0.0088	0.0156	0.0113	0.0064	0.0063	0.0205
3 20 10	(10, 0, ..., 0)	0.0576	0.0678	0.0558	0.0630	0.0180	0.0835	0.0386
	(0, ..., 0, 10)	0.1252	0.2956	0.3000	0.2964	0.2941	0.4551	0.2260
	(2, 0, 2, 0, 2, 0, 0, 2, 0, 2)	0.1058	0.1392	0.1577	0.1472	0.1307	0.1813	0.1221
15	(5, 0, ..., 0)	0.0176	0.0205	0.0327	0.0252	0.0160	0.0168	0.0340
	(0, ..., 0, 5)	0.0698	0.0761	0.0898	0.0817	0.0705	0.0880	0.0707
	(0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1)	0.0647	0.0557	0.0691	0.0610	0.0505	0.0616	0.0585
50 20	(30, 0, ..., 0)	0.0328	0.0220	0.0159	0.0198	0.0241	0.0304	0.0143
	(0, ..., 0, 30)	0.0569	0.1774	0.1784	0.1774	0.1771	0.2374	0.1170
	(3, 0, 3, 0, ..., 3, 0)	0.0468	0.0703	0.0789	0.0736	0.0668	0.0854	0.0571

Table 5. (Continued).

k n m	Scheme	$\hat{\beta}_{MLE}$	$\hat{\beta}_{SQR}$	$\hat{\beta}_{PRE}$	$\hat{\beta}_{LIN}$ λ=-0.5	$\hat{\beta}_{LIN}$ λ=0.5	$\hat{\beta}_{GE}$ λ=-0.5	$\hat{\beta}_{GE}$ λ=0.5
3 50 30	(20, 0, ..., 0)	0.0248	0.0023	0.0078	0.0043	0.0003	0.0000	0.0081
	(0, ..., 0, 20)	0.0344	0.0581	0.0643	0.0605	0.0557	0.0687	0.0442
	(2, 0, 0, 2, 0, 0, ..., 2, 0, 0)	0.0304	0.0336	0.0398	0.0359	0.0312	0.0379	0.0310
5 20 10	(10, 0, ..., 0)	0.0775	0.0399	0.0519	0.0446	0.0353	0.0441	0.0478
	(0, ..., 0, 10)	0.1274	0.4603	0.4259	0.4358	0.4993	0.8563	0.3760
	(2, 0, 2, 0, 2, 0, 0, 2, 0, 2)	0.1060	0.1928	0.2046	0.1972	0.1880	0.2882	0.1596
15	(5, 0, ..., 0)	0.0565	0.0204	0.0101	0.0160	0.0244	0.0346	0.0319
	(0, ..., 0, 5)	0.0704	0.1454	0.1545	0.1490	0.1414	0.1909	0.1075
	(0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1)	0.0642	0.0933	0.1050	0.0980	0.0885	0.1163	0.0777
50 20	(30,0,...,0)	0.0318	0.0291	0.0242	0.0273	0.0308	0.0379	0.0165
	(0,...,0,30)	0.0577	0.2236	0.2167	0.2194	0.2286	0.3192	0.1479
	(3,0,3,0,...,3,0)	0.0465	0.0681	0.0762	0.0712	0.0649	0.0863	0.0576
50 30	(20, 0, ..., 0)	0.0245	0.0074	0.0128	0.0094	0.0055	0.0067	0.0050
	(0, ..., 0, 20)	0.0349	0.0971	0.1010	0.0986	0.0956	0.1215	0.0632
	(2, 0, 0, 2, 0, 0, ..., 2, 0, 0)	0.0303	0.0473	0.0530	0.0494	0.0451	0.0561	0.0374

Table 6. MSL of $\hat{\beta}(.)$ when $(a, b) = (0.5, 3.0)$.

k n m	Scheme	$\hat{\beta}_{MLE}$	$\hat{\beta}_{SQR}$	$\hat{\beta}_{PRE}$	$\hat{\beta}_{LIN}$ λ=-0.5	$\hat{\beta}_{LIN}$ λ=0.5	$\hat{\beta}_{GE}$ λ=-0.5	$\hat{\beta}_{GE}$ λ=0.5
1 20 10	(10, 0, ..., 0)	0.0455	0.0324	0.0341	0.0340	0.0313	0.0212	0.0324
	(0, ..., 0, 10)	0.0584	0.0400	0.0417	0.0419	0.0387	0.0229	0.0391
	(2, 0, 2, 0, 2, 0, 0, 2, 0, 2)	0.0522	0.0362	0.0378	0.0378	0.0350	0.0214	0.0356
15	(5, 0, ..., 0)	0.0286	0.0211	0.0219	0.0218	0.0206	0.0165	0.0218
	(0, ..., 0, 5)	0.0273	0.0210	0.0215	0.0215	0.0205	0.0145	0.0215
	(0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1)	0.0286	0.0218	0.0226	0.0225	0.0213	0.0163	0.0223
50 20	(30, 0, ..., 0)	0.0153	0.0131	0.0135	0.0134	0.0129	0.0114	0.0133
	(0, ..., 0, 30)	0.0192	0.0179	0.0184	0.0183	0.0175	0.0169	0.0176
	(3, 0, 3, 0, ..., 3, 0)	0.0173	0.0159	0.0163	0.0162	0.0155	0.0146	0.0157
50 30	(20, 0, ..., 0)	0.0112	0.0100	0.0102	0.0102	0.0099	0.0091	0.0101
	(0, ..., 0, 20)	0.0104	0.0093	0.0095	0.0094	0.0092	0.0086	0.0095
	(2, 0, 0, 2, 0, 0, ..., 2, 0, 0)	0.0111	0.0101	0.0103	0.0102	0.0099	0.0093	0.0102
3 20 10	(10,0,...,0)	0.0483	0.0625	0.0655	0.0642	0.0313	0.0533	0.0291
	(0,...,0,10)	0.0596	0.1142	0.1100	0.1116	0.1166	0.4118	0.1159
	(2,0,2,0,2,0,0,2,0,2)	0.0756	0.0657	0.0675	0.0679	0.0633	0.1054	0.0588
15	(5, 0, ..., 0)	0.0229	0.0217	0.0224	0.0223	0.0211	0.0198	0.0207
	(0, ..., 0, 5)	0.0270	0.0296	0.0305	0.0304	0.0287	0.0336	0.0278
	(0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1)	0.0252	0.0265	0.0275	0.0273	0.0258	0.0281	0.0249
50 20	(30, 0, ..., 0)	0.0124	0.0115	0.0118	0.0117	0.0113	0.0260	0.0110
	(0, ..., 0, 30)	0.0199	0.0359	0.0346	0.0352	0.0365	0.0647	0.0325
	(3, 0, 3, 0, ..., 3, 0)	0.0161	0.0211	0.0214	0.0214	0.0208	0.0979	0.0188
50 30	(20, 0, ..., 0)	0.0088	0.0086	0.0087	0.0087	0.0085	0.0081	0.0084
	(0, ..., 0, 20)	0.0111	0.0129	0.0131	0.0131	0.0128	0.0149	0.0122
	(2, 0, 0, 2, 0, 0, ..., 2, 0, 0)	0.0100	0.0109	0.0111	0.0110	0.0108	0.0116	0.0104
5 20 10	(10, 0, ..., 0)	0.0343	0.0367	0.0394	0.0383	0.3086	0.0303	0.0302
	(0, ..., 0, 10)	0.0609	0.2187	0.1806	0.1852	0.0353	0.1250	0.3229
	(2, 0, 2, 0, 2, 0, 0, 2, 0, 2)	0.0479	0.0979	0.0952	0.0968	0.0988	0.2575	0.0872
15	(5, 0, ..., 0)	0.0219	0.0248	0.0256	0.0254	0.0411	0.0263	0.0224
	(0, ..., 0, 5)	0.0270	0.0415	0.0414	0.0417	0.0242	0.0678	0.0377
	(0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1)	0.0245	0.0332	0.0337	0.0338	0.0326	0.0444	0.0296
50 20	(30, 0, ..., 0)	0.0118	0.0120	0.0124	0.0122	0.0118	0.0104	0.0107
	(0, ..., 0, 30)	0.0204	0.0555	0.0500	0.0522	0.0598	0.1261	0.0479
	(3, 0, 3, 0, ..., 3, 0)	0.0159	0.0260	0.0261	0.0262	0.0258	0.0343	0.0210
50 30	(20, 0, ..., 0)	0.0085	0.0089	0.0091	0.0090	0.0088	0.0088	0.0084
	(0, ..., 0, 20)	0.0112	0.0166	0.0165	0.0166	0.0166	0.0227	0.0148
	(2, 0, 0, 2, 0, 0, ..., 2, 0, 0)	0.0098	0.0123	0.0124	0.0124	0.0122	0.0142	0.0111

Table 7. Bias of $\hat{\beta}(.)$ when $(a, b) = (1.5, 2.5)$.

k n m	Scheme	$\hat{\beta}_{\text{MLE}}$	$\hat{\beta}_{\text{SQR}}$	$\hat{\beta}_{\text{PRE}}$	$\hat{\beta}_{\text{LIN}}$		$\hat{\beta}_{\text{GE}}$	
					λ=-1	λ=1	λ=-1	λ=1
1 20 10	(10, 0, ..., 0)	0.1282	0.0059	0.0217	0.0269	0.0219	0.2475	0.0609
	(0, ..., 0, 10)	0.1720	0.0187	0.0439	0.0506	0.0245	0.3419	0.0930
	(2, 0, 2, 0, 2, 0, 0, 2, 0, 2)	0.1538	0.0238	0.0279	0.0328	0.0368	0.3164	0.0777
15	(5, 0, ..., 0)	0.0997	0.0032	0.0162	0.0188	0.0190	0.1737	0.0441
	(0, ..., 0, 5)	0.0901	0.0022	0.0082	0.0058	0.0020	0.1432	0.0419
	(0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1)	0.0978	0.0036	0.0121	0.0148	0.0161	0.2386	0.0470
50 20	(30, 0, ..., 0)	0.0537	0.0103	0.0065	0.0090	0.0112	0.0523	0.0252
	(0, ..., 0, 30)	0.0786	0.0417	0.0570	0.0587	0.0217	0.0964	0.0571
	(3, 0, 3, 0, ..., 3, 0)	0.0692	0.0239	0.0380	0.0394	0.0262	0.0647	0.0452
50 30	(20, 0, ..., 0)	0.0477	0.0084	0.0018	0.0070	0.0100	0.0427	0.0236
	(0, ..., 0, 20)	0.0384	0.0025	0.0006	0.0028	0.0041	0.0151	0.0190
	(2, 0, 0, 2, 0, 0, ..., 2, 0, 0)	0.0452	0.0050	0.0145	0.0151	0.0094	0.0221	0.0258
3 20 10	(10, 0, ..., 0)	0.1132	0.0336	0.0511	0.0533	0.0180	0.1637	0.0583
	(0, ..., 0, 10)	0.1781	0.4787	0.4745	0.4685	0.4828	1.6010	0.3004
	(2, 0, 2, 0, 2, 0, 0, 2, 0, 2)	0.1506	0.2391	0.2628	0.2668	0.2048	0.6352	0.1733
15	(5, 0, ..., 0)	0.0820	0.0012	0.0222	0.0265	0.0160	0.0070	0.0550
	(0, ..., 0, 5)	0.0993	0.1176	0.1369	0.1394	0.0949	0.2340	0.0971
	(0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1)	0.0921	0.0889	0.1078	0.1103	0.0678	0.1525	0.0820
50 20	(30, 0, ..., 0)	0.0467	0.0203	0.0150	0.0154	0.0290	0.1139	0.0214
	(0, ..., 0, 30)	0.0810	0.3094	0.3023	0.2993	0.3205	0.8623	0.1736
	(3, 0, 3, 0, ..., 3, 0)	0.0665	0.1371	0.1475	0.1481	0.1243	0.3324	0.1173
50 30	(20, 0, ..., 0)	0.0353	0.0070	0.0107	0.0100	0.0007	0.0210	0.0171
	(0, ..., 0, 20)	0.0489	0.0962	0.1044	0.1049	0.0865	0.2194	0.0634
	(2, 0, 0, 2, 0, 0, ..., 2, 0, 0)	0.0434	0.0589	0.0676	0.0680	0.0680	0.1173	0.0895
5 20 10	(10, 0, ..., 0)	0.1104	0.0835	0.1005	0.1023	0.0645	0.1572	0.0743
	(0, ..., 0, 10)	0.1813	0.8279	0.7312	0.6900	1.2338	3.1640	0.5310
	(2, 0, 2, 0, 2, 0, 0, 2, 0, 2)	0.1509	0.3869	0.3898	0.3854	0.3833	1.2412	0.2465
15	(5, 0, ..., 0)	0.0805	0.0242	0.0453	0.0494	0.0032	0.0641	0.0669
	(0, ..., 0, 5)	0.1002	0.2530	0.2610	0.2608	0.2406	0.6873	0.1563
	(0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1)	0.0914	0.1731	0.1873	0.1884	0.1545	0.4320	0.1170
50 20	(30, 0, ..., 0)	0.0452	0.0248	0.0327	0.0330	0.0241	0.0961	0.0281
	(0, ..., 0, 30)	0.0822	0.4570	0.4194	0.4089	0.5469	1.3480	0.2529
	(3, 0, 3, 0, ..., 3, 0)	0.0662	0.1789	0.1852	0.1848	0.1708	0.4684	0.1080
50 30	(20, 0, ..., 0)	0.0349	0.0155	0.0064	0.0058	0.0168	0.0311	0.0185
	(0, ..., 0, 20)	0.0497	0.1792	0.1808	0.1803	0.1769	0.4664	0.0984
	(2, 0, 0, 2, 0, 0, ..., 2, 0, 0)	0.0431	0.0976	0.1046	0.1048	0.0895	0.2321	0.0611

6. Real Life Data

In this example, we consider a real life data set to illustrate the proposed method and verify how our estimators work in practice. The validity of the IW model is checked using Kolmogrov-Smirnov $(K-S)$ test, as well as Anderson-Darling $(A-D)$ and chi-square tests. The data set for this application is provided by the Greater Amman Municipality in Jordan. The data consist of 64 readings that demonstrate the percentage of asphalt content in hot mix asphalt samples taken from highway pavement projects in 2012. Percentage of asphalt content is one of the main elements of a hot mix asphalt sample characteristics that has a direct effect on the quality and durability of the pavement. That is why this data is used in this example.

$(4.45, 4.82)$	$(4.69, 4.79)$	$(4.95, 4.87)$	$(4.29, 4.70)$	$(4.87, 4.54)$	$(4.87, 4.73)$
$(4.86, 4.26)$	$(4.29, 4.54)$	$(4.72, 4.62)$	$(4.54, 4.73)$	$(4.52, 4.74)$	$(4.58, 4.93)$
$(4.98, 4.28)$	$(4.61, 4.35)$	$(4.65, 4.85)$	$(4.70, 4.70)$	$(4.87, 4.98)$	$(4.46, 4.66)$
$(4.87, 4.44)$	$(4.86, 4.60)$	$(4.77, 4.58)$	$(4.82, 5.08)$	$(4.73, 4.62)$	$(5.11, 4.89)$
$(4.84, 4.76)$	$(5.04, 4.88)$	$(4.75, 4.74)$	$(4.80, 4.77)$	$(4.72, 4.72)$	$(4.77, 4.53)$
$(4.51, 4.59)$	$(4.70, 4.82)$				

Table 8. MSL of $\hat{\beta}(.)$ when $(a, b) = (1.5, 2.5)$.

k n m	Scheme	$\hat{\beta}_{MLE}$	$\hat{\beta}_{SQR}$	$\hat{\beta}_{PRE}$	$\hat{\beta}_{LIN}$ λ=-1	$\hat{\beta}_{LIN}$ λ=1	$\hat{\beta}_{GE}$ λ=-1	$\hat{\beta}_{GE}$ λ=1
1 20 10	(10, 0, ..., 0)	0.0922	0.0541	0.0552	0.0601	0.0525	0.3461	0.0652
	(0, ..., 0,10)	0.1183	0.0638	0.0636	0.0695	0.0643	2.3110	0.0811
	(2, 0, 2, 0, 2, 0, 0, 2, 0, 2)	0.1057	0.0589	0.0592	0.0645	0.0587	1.0081	0.0732
15	(5, 0, ..., 0)	0.0579	0.0378	0.0386	0.0410	0.0365	0.0293	0.0428
	(0, ..., 0, 5)	0.0553	0.0372	0.0375	0.0395	0.0361	0.0191	0.0422
	(0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1)	0.0580	0.0388	0.0396	0.0419	0.0374	0.0231	0.0443
50 20	(30, 0, ..., 0)	0.0310	0.0251	0.0256	0.0265	0.0240	0.0119	0.0265
	(0, ..., 0, 30)	0.0389	0.0336	0.0346	0.0360	0.0317	0.0247	0.0344
	(3, 0, 3, 0, ..., 3, 0)	0.0351	0.0301	0.0308	0.0320	0.0285	0.0198	0.0309
50 30	(20, 0, ..., 0)	0.0228	0.0197	0.0194	0.0202	0.0179	0.0116	0.0202
	(0, ..., 0, 20)	0.0211	0.0181	0.0184	0.0188	0.0175	0.0113	0.0189
	(2, 0, 0, 2, 0, 0, ..., 2, 0, 0)	0.0225	0.0195	0.0198	0.0203	0.0188	0.0129	0.0212
3 20 10	(10, 0, ..., 0)	0.0736	0.0590	0.0615	0.0661	0.0548	0.0365	0.0594
	(0, ..., 0, 10)	0.1207	0.2064	0.2021	0.1967	0.2102	6.0660	0.1653
	(2, 0, 2, 0, 2, 0, 0, 2, 0, 2)	0.0979	0.1191	0.1231	0.1297	0.1050	0.7218	0.1026
15	(5, 0, ..., 0)	0.0464	0.0407	0.0420	0.0441	0.0380	0.0299	0.0407
	(0, ..., 0, 5)	0.0546	0.0551	0.0571	0.0599	0.0504	0.0979	0.0524
	(0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1)	0.0511	0.0432	0.0515	0.0540	0.0458	0.0694	0.0478
50 20	(30, 0, ..., 0)	0.0252	0.0224	0.0229	0.0234	0.0216	0.0138	0.0224
	(0, ..., 0, 30)	0.0403	0.0701	0.0675	0.0658	0.0760	0.4081	0.0588
	(3, 0, 3, 0, ..., 3, 0)	0.0326	0.0413	0.0418	0.0425	0.0397	0.1074	0.0365
50 30	(20, 0, ..., 0)	0.0179	0.0169	0.0171	0.0175	0.0163	0.0146	0.0168
	(0, ..., 0, 20)	0.0225	0.0254	0.0257	0.0261	0.0245	0.0467	0.0238
	(2, 0, 0, 2, 0, 0, ..., 2, 0, 0)	0.0204	0.0215	0.0218	0.0222	0.0222	0.0303	0.0303
5 20 10	(10, 0, ..., 0)	0.0695	0.0683	0.0717	0.0764	0.0621	0.0680	0.0617
	(0, ..., 0, 10)	0.1233	0.4137	0.3396	0.2760	1.2181	7.4301	0.3478
	(2, 0, 2, 0, 2, 0, 0, 2, 0, 2)	0.0970	0.1841	0.1770	0.1725	0.1949	3.3457	0.1404
15	(5, 0, ..., 0)	0.0444	0.0474	0.0488	0.0508	0.0440	0.0802	0.0437
	(0, ..., 0, 5)	0.0546	0.0793	0.0792	0.0797	0.0770	0.4107	0.0675
	(0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1)	0.0497	0.0637	0.0647	0.0663	0.0599	0.2143	0.0555
50 20	(30, 0, ..., 0)	0.0239	0.0234	0.0239	0.0245	0.0224	0.0217	0.0220
	(0, ..., 0, 30)	0.0413	0.1104	0.0968	0.0888	0.1891	1.0115	0.0886
	(3, 0, 3, 0, ..., 3, 0)	0.0322	0.0515	0.0507	0.0508	0.0517	0.1937	0.0416
50 30	(20, 0, ..., 0)	0.0172	0.0177	0.0180	0.0183	0.0171	0.0205	0.0170
	(0, ..., 0, 20)	0.0228	0.0329	0.0325	0.0325	0.0332	0.1061	0.0286
	(2, 0, 0, 2, 0, 0, ..., 2, 0, 0)	0.0199	0.0245	0.0246	0.0249	0.0239	0.0506	0.0221

We fit the IW distribution based on $\alpha = 0.209$ and $\beta = 29.083$. We observe that $K - S = 0.0864$ with $p_{value} = 0.8177$, $A - D = 0.3621$ and chi-square distance $= 0.6468$ with a corresponding $p_{value} = 0.98576$. This indicates that the IW model provides a good fit. The initial estimates for the $MLEs$ are chosen by using pseudo complete estimates of the $MLEs$. We group the data into 32 sets with 2 items in each. We modify the data to consider four types of censoring as follows:

Case 1: The complete data set where $k = 1,\ R = (0, 0, \ldots, 0, 0)$.

Case 2: Progressive first-failure censoring, $k = 2,\ R = (12, 0, \ldots, 0, 0)$.

Case 3: First failure censoring, $k = 2,\ R = (0, 0, 0, \ldots, 0)$.

Case 4: Progressive type-II, $k = 2,\ R = (0, 0, \ldots, 0, 24)$.

The Modified data sets are provided in Table 9. The estimates of α and β based on different estimation methods are provided in Table 10.

It is quite clear that all the Bayes estimates of the shape and scale parameters are quite close to each other. It is of great importance to notice through this analysis that the estimates based on progressively first-failure are comparable with the values of the estimates based on progressively type-II censored samples and they are very close to those of the complete data set.

The implication of this inference with regard to the real data used reflects on construction limit bounds for the optimum asphalt content for the duration of a certain project executed by a certain contractor company. This helps in measuring the acceptance of the asphalt mix prepared daily in the mixing plant with regard to the required specifications, thus judging the proficiency of that contractor company in a shorter period of time.

Table 9. Progressive first-failure censored samples for the percentage of asphalt content in hot mix samples.

case	n	m	censored data											
1	64	65	4.26	4.28	4.29	4.29	4.35	4.44	4.45	4.46	4.51	4.52	4.53	4.54
			4.54	4.54	4.58	4.58	4.59	4.60	4.61	4.62	4.62	4.65	4.66	4.69
			4.70	4.70	4.70	4.70	4.72	4.72	4.72	4.73	4.73	4.73	4.74	4.74
			4.75	4.76	4.77	4.77	4.77	4.79	4.80	4.82	4.82	4.82	4.84	4.85
			4.86	4.86	4.87	4.87	4.87	4.87	4.87	4.88	4.89	4.93	4.95	4.98
			4.98	5.04	5.08	5.11								
2	32	20	4.35	4.44	4.45	4.46	4.51	4.53	4.58	4.60	4.62	4.65	4.70	4.70
			4.72	4.74	4.76	4.77	4.82	4.87	4.88	4.89				
3	32	32	4.45	4.69	4.87	4.29	4.54	4.73	4.26	4.29	4.62	4.54	4.52	4.58
			4.28	4.35	4.65	4.70	4.87	4.46	4.44	4.60	4.58	4.82	4.62	4.89
			4.76	4.88	4.74	4.77	4.72	4.53	4.51	4.70				
4	64	40	4.26	4.28	4.29	4.29	4.35	4.44	4.45	4.46	4.52	4.54	4.54	4.54
			4.58	4.60	4.61	4.62	4.65	4.66	4.69	4.70	4.70	4.70	4.72	4.73
			4.73	4.74	4.79	4.82	4.85	4.86	4.86	4.87	4.87	4.87	4.87	4.87
			4.93	4.95	4.98	4.98								

Table 10. The corresponding estimates.

Method	$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{SQR}$	$\hat{\alpha}_{LIN}$	$\hat{\alpha}_{GE}$	$\hat{\alpha}_{PRE}$	$\hat{\beta}_{MLE}$	$\hat{\beta}_{SQR}$	$\hat{\beta}_{LIN}$	$\hat{\beta}_{GE}$	$\hat{\beta}_{PRE}$
case 1	0.2172	0.2180	0.2180	0.2173	0.2181	22.5324	25.9744	23.3021	22.5324	25.8221
case 2	0.2137	0.2152	0.2152	0.2138	0.2153	22.5061	27.6015	21.8463	22.5061	27.3485
case 3	0.2159	0.2196	0.2196	0.2178	0.2196	18.2935	27.7468	22.0053	18.2936	26.1894
case 4	0.2174	0.2195	0.2193	0.2174	0.2194	22.0674	27.5370	21.6614	22.0675	27.0981

7. Conclusions and Recommendations

In the past few years progressive censoring has received a great attention by many researchers. This is due to its advantages in reducing the cost and time of the tests. Moreover, the availability of high speed computing resources enhance the focus on progressive censoring. In this article we have considered the MLE and the Bayesians (SQR, LIN, GE, PRE) to estimate the unknown parameters of the IW distribution when data under consideration are progressively first-failure censored data.

The Bayes estimates of the unknown parameters are computed based on symmetrical and asymmetrical loss functions. The Bayes estimates do not have explicit forms, therefore, we use the Lindley's approximation method under the assumption of gamma priors. We study and compare the performance of the $MLEs$ and Bayesian estimates based on extensive simulations. We observe that in most of the cases the Bayes estimates have the smallest *Bias* and smallest *MSE* relative to the *MLEs*. Overall, we suggest using the Bayes estimates for estimating the parameters of the *IW* distribution based on progressively first-failure censoring data.

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